Journal of Global Optimization **29:** 125–155, 2004.

Trilinear Monomials with Mixed Sign Domains: Facets of the Convex and Concave Envelopes

CLIFFORD A. MEYER and CHRISTODOULOS A. FLOUDAS
Department of Chemical Engineering, Princeton University, Princeton, NJ 08544, USA (email: floudas@titan.princeton.edu)

(Received 14 January 2003; accepted in revised form 9 May 2003)

Abstract. Convex underestimators of nonconvex functions, frequently used in deterministic global optimization algorithms, strongly influence their rate of convergence and computational efficiency. A good convex underestimator is as tight as possible and introduces a minimal number of new variables and constraints. Multilinear monomials over a coordinate aligned hyper-rectangular domain are known to have polyhedral convex envelopes which may be represented by a finite number of facet inducing inequalities. This paper describes explicit expressions defining the facets of the convex and concave envelopes of trilinear monomials over a box domain with bounds of opposite signs for at least one variable. It is shown that the previously used approximations based on the recursive use of the bilinear construction rarely yield the convex envelope itself.

Key words: Concave envelope, Convex envelope, Convex underestimators, Trilinear monomial

1. Introduction

Convex underestimators of nonconvex functions are frequently used in deterministic global optimization algorithms such as the αBB algorithm of Adjiman et al. (1998a, b). A good convex underestimator is as tight as possible and requires a minimal number of new variables and constraints. As the underestimator becomes tighter, the number of nodes in the branch and bound tree decreases. A reduction in the number of variables and constraints in the representation of the convex underestimator reduces the computational effort to process a node in the tree.

The pointwise supremum of all convex underestimators of a lower semi-continuous function $f : \mathbf{x} \ni x \to \Re$ is referred to as the *convex envelope* of f over the domain $\mathbf{x}$. Similarly, the *concave envelope* of f is the pointwise infimum of all concave overestimators of f on $\mathbf{x}$. Rikun (1997) has shown that multilinear monomials over a hyper-rectangular domain have polyhedral convex and concave envelopes which may be represented by a finite number of facet inducing inequalities. Although the facets of individual bilinear monomials over a rectangular domain have been determined by McCormick (1976), and Al-Khayyal and Falk (1983), the convex envelopes of higher order multilinear terms have been approximated, based on the recursive use of the bilinear case (Hamed, 1991; Maranas and Floudas, 1995; Ryoo and Sahinidis, 2001). A formula for the maximum distance separating the convex envelope and a bilinear function was determined by Androulakis (1995), also see Floudas (2000). Rikun (1997) postulated an equation which, in some instances, may be used to define some facets of a multilinear function. Tawarmalani and Sahinidis (2002) have shown in their Theorem 9 that the convex envelope of a multilinear function f, over a Cartesian product of polytopes, matches f only on the faces over which f is linear. However, the explicit form of the convex envelope facets is not given in this paper.

Further results on linear relaxations of multilinear functions have been obtained in the context of 0-1 programming. These include the contributions of Balas and Mazzola (1984), Crama (1989), Glover and Woolsey (1974), Hammer et al. (1984), Hammer and Rudeanu (1968), and Hansen and Simeone (1990). Crama (1993) has characterized certain situations in which the "standard linearization" provides the convex envelope of a multilinear function over the unit hypercube. More recently, Sherali (1997) has derived the convex envelope for additional classes of 0-1 multilinear function over the unit hypercube and special discrete sets, but not for the general case.

Explicit expressions defining the facets of the convex and concave envelopes of a trilinear monomial over a hyper-rectangle, are derived in this paper. These expressions are dependent on the signs on the box bounds. We consider in this paper the cases where at least one variable has a negative lower bound and a positive upper bound. In Meyer and Floudas (2003) we present the explicit expressions of the convex and concave envelope facets for the cases of positive or negative domains.

The convex hull of the graph of the trilinear monomial is defined as follows:

$$\mathcal{C}_3(\mathbf{x}) := \operatorname{conv}(\{(x, x_1x_2x_3) : x \in \mathbf{x}\}),$$

where $\mathbf{x} := [\underline{x}_1, \overline{x}_1] \times [\underline{x}_2, \overline{x}_2] \times [\underline{x}_3, \overline{x}_3]$, and $\operatorname{conv}(\cdot)$, denotes the convex hull.

$\underline{\mathcal{C}}_3(\mathbf{x})$ and $\overline{\mathcal{C}}_3(\mathbf{x})$, denote the convex hulls of the epigraph and hypograph of $x_1x_2x_3, (x \in \mathbf{x})$:

$$\underline{\mathcal{C}}_3(\mathbf{x}) := \operatorname{conv}\big(\{(x, w) : w \geq x_1x_2x_3, x \in \mathbf{x}, (x, w) \in \Re^3 \times \Re\}\big)$$
$$\overline{\mathcal{C}}_3(\mathbf{x}) := \operatorname{conv}\big(\{(x, w) : w \leq x_1x_2x_3, x \in \mathbf{x}, (x, w) \in \Re^3 \times \Re\}\big).$$

The nonvertical facets of $\underline{\mathcal{C}}_3(\mathbf{x})$ and $\overline{\mathcal{C}}_3(\mathbf{x})$ are respectively the facets of the convex and concave envelopes of $x_1x_2x_3$ over $\mathbf{x}$. It follows from Proposition 1.1 that the convex hull of the graph of a trilinear monomial is polyhedral.

PROPOSITION 1.1 *Consider the trilinear monomial in* $\Re^3$, $x_1x_2x_3$, *and let* $\mathcal{V} = \{x^1, \ldots, x^8\}$ *be the set of vertices of the hyper-rectangle* $[\underline{x}, \overline{x}]$. *Then the convex hull,* $\mathcal{C}_3$, *is the convex hull of the vertices,*

$$\mathrm{conv}\big(\{(x^k, x_1^k x_2^k x_3^k) : x^k \in \mathcal{V}\}\big).$$

A proof of this proposition is provided in the appendix.

We assume throughout that the lower bounds are strictly less than the upper bounds, $\underline{x}_i < \overline{x}_i$ for all x_i, hence by Carathéodory's Theorem (Rockafellar, 1970) each facet of $\mathcal{C}_3(\mathbf{x})$ can be written as a convex combination of four points from the set $\{(x, \prod_{i=1}^{3} x_i) : x \in \mathcal{V}\}$. When the upper and lower bounds on a single variable are identical, the inequalities induced by the facets of Al-khayyal and Falk (1983) for bilinear monomials may be used to represent the convex hull.

Each facet is associated with a cell of a triangulation of the domain. Once the correct triangulation is known, each facet defining hyperplane is determined by a set of four linear equations.

2. Convex Envelope Facet Classes

The four dimensional systems of linear equations defining the facets of the convex hull are each associated with a simplex with vertices from the vertex set of $\mathbf{x}$. Four types of simplex may be constructed from the vertices of the hyper-rectangle. These types may be characterized, respectively, as having zero, one, two or three facets which are contained in the facets of the hyper-rectangle $\mathbf{x}$. The following four lemmas describe the form of the hyperplanes for each of these four types of simplex and the conditions under which the hyperplanes are facets of $\underline{\mathcal{C}}_3(\mathbf{x})$ and $\overline{\mathcal{C}}_3(\mathbf{x})$. For $\underline{\mathcal{C}}_3(\mathbf{x})$ and $\overline{\mathcal{C}}_3(\mathbf{x})$ the statements and the proofs are similar, the only difference being the reversal of $\leq$ to $\geq$ in the inequalities. In the lemmas the $\overline{\mathcal{C}}_3(\mathbf{x})$ case is indicated in parentheses. In each of these lemmas we let $\{\chi, \psi, \omega\}$ denote a permutation of $\{x_1, x_2, x_3\}$, and use superscripts A and B to denote upper and lower bounds interchangeably. If x_i^A corresponds to a lower bound on x_i then x_i^B corresponds to the upper bound on the same variable. A may be used to indicate the upper bound on one variable x_i while indicating the lower bound on another variable x_j, $j \neq i$. For example, the substitutions $\chi^A \leftarrow \underline{x}_2$, $\psi^A \leftarrow \overline{x}_1$ and $\omega^A \leftarrow \underline{x}_3$ imply that $\chi^B = \overline{x}_2$, $\psi^B = \underline{x}_1$ and $\omega^B = \overline{x}_3$. The proofs of these lemmas are provided in the appendix.

LEMMA 2.1. *Under the conditions,*

$$\chi^A\psi^A\omega^A + \chi^B\psi^B\omega^A + \chi^B\psi^A\omega^B \leq (\geq)\ 2\chi^B\psi^A\omega^A + \chi^A\psi^B\omega^B, \tag{2.1}$$

$$\chi^A\psi^A\omega^A + \chi^B\psi^B\omega^A + \chi^A\psi^B\omega^B \leq (\geq)\ 2\chi^A\psi^B\omega^A + \chi^B\psi^A\omega^B, \tag{2.2}$$

$$\chi^A\psi^A\omega^A + \chi^B\psi^A\omega^B + \chi^A\psi^B\omega^B \leq (\geq)\ 2\chi^A\psi^A\omega^B + \chi^B\psi^B\omega^A, \tag{2.3}$$

$$\chi^B\psi^B\omega^A + \chi^B\psi^A\omega^B + \chi^A\psi^B\omega^B \leq (\geq)\ 2\chi^B\psi^B\omega^B + \chi^A\psi^A\omega^A, \tag{2.4}$$

the hyperplane,

$$w = \theta_\chi\chi + \theta_\psi\psi + \theta_\omega\omega + \theta_c,$$

where,

$$\begin{aligned}
\theta_\chi &= \frac{1}{2}(\chi^A\psi^B\omega^B + \chi^A\psi^A\omega^A - \chi^B\psi^B\omega^A - \chi^B\psi^A\omega^B)/(\chi^A - \chi^B)\\
\theta_\psi &= \frac{1}{2}(\chi^B\psi^A\omega^B + \chi^A\psi^A\omega^A - \chi^B\psi^B\omega^A - \chi^A\psi^B\omega^B)/(\psi^A - \psi^B)\\
\theta_\omega &= \frac{1}{2}(\chi^B\psi^B\omega^A + \chi^A\psi^A\omega^A - \chi^B\psi^A\omega^B - \chi^A\psi^B\omega^B)/(\omega^A - \omega^B)\\
\theta_c &= \chi^A\psi^A\omega^A - \theta_\chi\chi^A - \theta_\psi\psi^A - \theta_\omega\omega^A
\end{aligned}$$

defines the affine hull of a facet of $\underline{\mathcal{C}}_3(\mathbf{x})(\overline{\mathcal{C}}_3(\mathbf{x}))$. *The projected vertices of the facet are*

$$\{(\chi^A, \psi^A, \omega^A), (\chi^B, \psi^B, \omega^A), (\chi^B, \psi^A, \omega^B), (\chi^A, \psi^B, \omega^B)\}.$$

LEMMA 2.2. *Under the conditions,*

$$\chi^B\psi^B\omega^A + \chi^B\psi^A\omega^B \leq (\geq)\ \chi^B\psi^A\omega^A + \chi^B\psi^B\omega^B, \tag{2.5}$$

$$\chi^B\psi^A\omega^A + \chi^A\psi^B\omega^B \leq (\geq)\ \chi^A\psi^B\omega^A + \chi^B\psi^A\omega^B, \tag{2.6}$$

$$\chi^B\psi^A\omega^A + \chi^A\psi^B\omega^B \leq (\geq)\ \chi^A\psi^A\omega^B + \chi^B\psi^B\omega^A, \tag{2.7}$$

$$2\chi^B\psi^A\omega^A + \chi^A\psi^B\omega^B \leq (\geq)\ \chi^A\psi^A\omega^A + \chi^B\psi^B\omega^A + \chi^B\psi^A\omega^B, \tag{2.8}$$

the following equation defines the affine hull of a facet of $\underline{\mathcal{C}}_3(\mathbf{x})$ $(\overline{\mathcal{C}}_3(\mathbf{x}))$:

$$\begin{aligned}
w = &\frac{\theta}{\chi^B - \chi^A}\chi + \chi^B\omega^A\psi + \chi^B\psi^A\omega\\
&+ \left(-\frac{\theta\chi^A}{\chi^B - \chi^A} - \chi^B\psi^B\omega^A - \chi^B\psi^A\omega^B + \chi^A\psi^B\omega^B\right),
\end{aligned}$$

where $\theta = \chi^B\psi^B\omega^A - \chi^A\psi^B\omega^B - \chi^B\psi^A\omega^A + \chi^B\psi^A\omega^B$.

The projected vertices of the facet are:

$$\{(\chi^B, \psi^B, \omega^A), (\chi^A, \psi^B, \omega^B), (\chi^B, \psi^A, \omega^B), (\chi^B, \psi^A, \omega^A)\}.$$

LEMMA 2.3. *Under the conditions,*

$$\chi^A\psi^A\omega^B + \chi^B\psi^A\omega^A \le (\ge)\ \chi^B\psi^A\omega^B + \chi^A\psi^A\omega^A \tag{2.9}$$

$$\chi^B\psi^A\omega^B + \chi^A\psi^B\omega^B \le (\ge)\ \chi^B\psi^B\omega^B + \chi^A\psi^A\omega^B \tag{2.10}$$

$$\chi^B\psi^A\omega^A + \chi^A\psi^B\omega^B \le (\ge)\ \chi^A\psi^B\omega^A + \chi^B\psi^A\omega^B \tag{2.11}$$

$$\chi^B\psi^A\omega^A + \chi^A\psi^B\omega^B \le (\ge)\ \chi^A\psi^A\omega^B + \chi^B\psi^B\omega^A \tag{2.12}$$

the equation,

$$w = \psi^A\omega^B\chi + \chi^A\omega^B\psi + \chi^B\psi^A\omega - \chi^A\psi^A\omega^B - \chi^B\psi^A\omega^B \tag{2.13}$$

defines the affine hull of a facet of $\underline{\mathcal{C}}_3(\mathbf{x})$ $(\overline{\mathcal{C}}_3(\mathbf{x}))$. *The projected vertices of the facet are:*

$$\{(\chi^A, \psi^B, \omega^B), (\chi^A, \psi^A, \omega^B), (\chi^B, \psi^A, \omega^B), (\chi^B, \psi^A, \omega^A)\}.$$

LEMMA 2.4. *Under the conditions,*

$$\chi^A\psi^A\omega^B + \chi^B\psi^A\omega^A \le (\ge)\ \chi^A\psi^A\omega^A + \chi^B\psi^A\omega^B, \tag{2.14}$$

$$\chi^A\psi^A\omega^B + \chi^A\psi^B\omega^A \le (\ge)\ \chi^A\psi^A\omega^A + \chi^A\psi^B\omega^B, \tag{2.15}$$

$$\chi^A\psi^B\omega^A + \chi^B\psi^A\omega^A \le (\ge)\ \chi^A\psi^A\omega^A + \chi^B\psi^B\omega^A, \tag{2.16}$$

$$\chi^A\psi^A\omega^B + \chi^B\psi^A\omega^A + \chi^A\psi^B\omega^A \le (\ge)\ 2\chi^A\psi^A\omega^A + \chi^B\psi^B\omega^B, \tag{2.17}$$

the hyperplane,

$$w = \psi^A\omega^A\chi + \chi^A\omega^A\psi + \chi^A\psi^A\omega - 2\chi^A\psi^A\omega^A \tag{2.18}$$

defines the affine hull of a facet of $\underline{\mathcal{C}}_3(\mathbf{x})$ $(\overline{\mathcal{C}}_3(\mathbf{x}))$. *The projected vertices of the facet are*

$$\{(\chi^A, \psi^A, \omega^A), (\chi^B, \psi^A, \omega^A), (\chi^A, \psi^B, \omega^A), (\chi^A, \psi^A, \omega^B)\}.$$

3. Main Results: Facets of the Convex Envelope

The description of the nonvertical facets of $\underline{\mathcal{C}}_3(\mathbf{x})$ depends on the signs of the bounds on $\mathbf{x}$. Here we give the facets for the cases where, for at least one variable, the lower bound is negative and the upper bound is positive. The symbols x, y, and z are used to denote a permutation of x_1, x_2 and x_3. In addition to the signs of the bounds, in some cases there are auxiliary inequalities that must be satisfied for the facets to apply.

3.1. CASE 1: $\underline{x} \geq 0,\ \underline{y} \geq 0,\ \underline{z} \leq 0,\ \overline{z} \geq 0$

The linear equalities defining the facets of $\underline{\mathcal{C}}_3(x)$ are:

$$w = \overline{y}\overline{z}x + \overline{x}\overline{z}y + \overline{x}\overline{y}z - 2\overline{x}\overline{y}\overline{z} \tag{3.19}$$

$$w = \overline{y}\underline{z}x + \underline{x}\overline{z}y + \underline{x}\overline{y}z - \underline{x}\overline{y}\underline{z} - \underline{x}\overline{y}\overline{z} \tag{3.20}$$

$$w = \overline{y}\underline{z}x + \underline{x}\underline{z}y + \underline{x}\underline{y}z - \underline{x}\overline{y}\underline{z} - \underline{x}\underline{y}\underline{z} \tag{3.21}$$

$$w = \underline{y}\overline{z}x + \overline{x}\underline{z}y + \overline{x}\underline{y}z - \overline{x}\underline{y}\overline{z} - \overline{x}\underline{y}\underline{z} \tag{3.22}$$

$$w = \underline{y}\underline{z}x + \overline{x}\underline{z}y + \underline{x}\underline{y}z - \overline{x}\underline{y}\underline{z} - \underline{x}\underline{y}\underline{z} \tag{3.23}$$

$$\begin{aligned} w =& \underline{y}\overline{z}x + \underline{x}\overline{z}y + \frac{\theta}{\overline{z} - \underline{z}} z \\ &+ \left(-\frac{\theta \underline{z}}{\overline{z} - \underline{z}} - \underline{x}\overline{y}\overline{z} - \overline{x}\underline{y}\overline{z} + \overline{x}\overline{y}\underline{z} \right), \end{aligned} \tag{3.24}$$

where $\theta = \underline{x}\overline{y}\overline{z} - \overline{x}\overline{y}\underline{z} - \underline{x}\underline{y}\overline{z} + \overline{x}\underline{y}\overline{z}$.

3.2. CASE 2: $\underline{x} \geq 0,\ \underline{y} \leq 0,\ \underline{z} \leq 0,\ \overline{y} \geq 0,\ \overline{z} \geq 0$

Mapping $\{x_1, x_2, x_3\}$ onto $\{x, y, z\}$ in such a way that the following relation applies,

$$\overline{x}\overline{y}\underline{z} + \underline{x}\underline{y}\overline{z} \leq \underline{x}\overline{y}\underline{z} + \overline{x}\underline{y}\overline{z},$$

the linear equalities defining the facets of $\underline{\mathcal{C}}_3(x)$ are:

$$w = \overline{y}\overline{z}x + \overline{x}\overline{z}y + \overline{x}\overline{y}z - 2\overline{x}\overline{y}\overline{z} \tag{3.25}$$

$$w = \underline{y}\underline{z}x + \overline{x}\underline{z}y + \overline{x}\underline{y}z - 2\overline{x}\underline{y}\underline{z} \tag{3.26}$$

$$w = \overline{y}\underline{z}x + \underline{x}\overline{z}y + \underline{x}\overline{y}z - \underline{x}\overline{y}\underline{z} - \underline{x}\overline{y}\overline{z} \tag{3.27}$$

$$w = \overline{y}\underline{z}x + \underline{x}\underline{z}y + \underline{x}\underline{y}z - \underline{x}\overline{y}\underline{z} - \underline{x}\underline{y}\underline{z} \tag{3.28}$$

$$\begin{aligned} w =& \underline{y}\overline{z}x + \frac{\theta_1}{\underline{y} - \overline{y}} y + \underline{x}\underline{y}z \\ &+ \left(-\frac{\theta_1 \overline{y}}{\underline{y} - \overline{y}} - \underline{x}\underline{y}\underline{z} - \overline{x}\underline{y}\overline{z} + \overline{x}\overline{y}\underline{z} \right), \end{aligned} \tag{3.29}$$

where $\theta_1 = \underline{x}\underline{y}\underline{z} - \overline{x}\overline{y}\underline{z} - \underline{x}\underline{y}\overline{z} + \overline{x}\underline{y}\overline{z}$

$$w = \underline{y}\overline{z}x + \underline{x}\overline{z}y + \frac{\theta_2}{\overline{z} - \underline{z}} z + \left(-\frac{\theta_2 \underline{z}}{\overline{z} - \underline{z}} - \overline{x}\underline{y}\overline{z} - \underline{x}\overline{y}\overline{z} + \overline{x}\overline{y}\underline{z} \right), \tag{3.30}$$

where $\theta_2 = \overline{x}\underline{y}\overline{z} - \overline{x}\overline{y}\underline{z} - \underline{x}\underline{y}\overline{z} + \underline{x}\overline{y}\overline{z}$.

3.3. CASE 3: $\underline{x} \le 0, \underline{y} \le 0, \underline{z} \le 0, \overline{x} \ge 0, \overline{y} \ge 0, \overline{z} \ge 0$

Map $\{x_1, x_2, x_3\}$ onto $\{x, y, z\}$. If the following conditions apply:

$$\overline{x}\overline{y}\underline{z} + \overline{x}\underline{y}\overline{z} + \underline{x}\overline{y}\overline{z} \le \underline{x}\underline{y}\underline{z} + 2\overline{x}\overline{y}\overline{z},$$
$$\underline{x}\underline{y}\underline{z} + \overline{x}\overline{y}\underline{z} + \overline{x}\underline{y}\overline{z} \le \underline{x}\overline{y}\overline{z} + 2\overline{x}\underline{y}\underline{z},$$
$$\underline{x}\underline{y}\underline{z} + \overline{x}\overline{y}\underline{z} + \underline{x}\overline{y}\overline{z} \le \overline{x}\underline{y}\overline{z} + 2\underline{x}\overline{y}\underline{z},$$
$$\underline{x}\underline{y}\underline{z} + \overline{x}\underline{y}\overline{z} + \underline{x}\overline{y}\overline{z} \le \overline{x}\overline{y}\underline{z} + 2\underline{x}\underline{y}\overline{z},$$

the linear equalities defining the facets of $\underline{\mathcal{C}}_3(x)$ are:

$$w = \overline{y}\underline{z}x + \underline{x}\underline{z}y + \underline{x}\overline{y}z - 2\underline{x}\overline{y}\underline{z}$$
$$w = \underline{y}\overline{z}x + \underline{x}\overline{z}y + \underline{x}\underline{y}z - 2\underline{x}\underline{y}\overline{z}$$
$$w = \overline{y}\overline{z}x + \overline{x}\overline{z}y + \overline{x}\overline{y}z - 2\overline{x}\overline{y}\overline{z}$$
$$w = \underline{y}\underline{z}x + \overline{x}\underline{z}y + \overline{x}\underline{y}z - 2\overline{x}\underline{y}\underline{z}$$
$$w = \theta_x x + \theta_y y + \theta_z z + \theta_c,$$

where

$$\theta_x = \frac{1}{2}(\underline{x}\overline{y}\overline{z} + \underline{x}\underline{y}\underline{z} - \overline{x}\overline{y}\underline{z} - \overline{x}\underline{y}\overline{z})/(\underline{x} - \overline{x}),$$
$$\theta_y = \frac{1}{2}(\overline{x}\underline{y}\overline{z} + \underline{x}\underline{y}\underline{z} - \overline{x}\overline{y}\underline{z} - \underline{x}\overline{y}\overline{z})/(\underline{y} - \overline{y}),$$
$$\theta_z = \frac{1}{2}(\overline{x}\overline{y}\underline{z} + \underline{x}\underline{y}\underline{z} - \overline{x}\underline{y}\overline{z} - \underline{x}\overline{y}\overline{z})/(\underline{z} - \overline{z}),$$
$$\theta_c = \underline{x}\underline{y}\underline{z} - \theta_x\underline{x} - \theta_y\underline{y} - \theta_z\underline{z}.$$

Otherwise if the following condition applies:

$$\overline{x}\overline{y}\underline{z} + \overline{x}\underline{y}\overline{z} + \underline{x}\overline{y}\overline{z} \ge \underline{x}\underline{y}\underline{z} + 2\overline{x}\overline{y}\overline{z},$$

the linear equalities defining the facets of $\underline{\mathcal{C}}_3(x)$ are:

$$w = \overline{y}\underline{z}x + \underline{x}\underline{z}y + \underline{x}\overline{y}z - 2\underline{x}\overline{y}\underline{z}$$
$$w = \underline{y}\overline{z}x + \underline{x}\overline{z}y + \underline{x}\underline{y}z - 2\underline{x}\underline{y}\overline{z}$$
$$w = \underline{y}\underline{z}x + \overline{x}\underline{z}y + \overline{x}\underline{y}z - 2\overline{x}\underline{y}\underline{z}$$

$$w = \frac{\theta_1}{\overline{x} - \underline{x}} x + \overline{x}\overline{z} y + \overline{x}\overline{y} z + \left(-\frac{\theta_1 \underline{x}}{\overline{x} - \underline{x}} - \overline{x}\overline{y}\underline{z} - \overline{x}\underline{y}\overline{z} + \underline{x}\underline{y}\underline{z} \right),$$

where

$$\theta_1 = \overline{x}\overline{y}\underline{z} - \underline{x}\underline{y}\underline{z} - \overline{x}\overline{y}\overline{z} + \overline{x}\underline{y}\overline{z}$$

$$w = \overline{y}\overline{z} x + \overline{x}\overline{z} y + \frac{\theta_2}{\overline{z} - \underline{z}} z + \left(-\frac{\theta_2 \underline{z}}{\overline{z} - \underline{z}} - \overline{x}\underline{y}\overline{z} - \underline{x}\overline{y}\overline{z} + \underline{x}\underline{y}\underline{z} \right),$$

where

$$\theta_2 = \overline{x}\underline{y}\overline{z} - \underline{x}\underline{y}\underline{z} - \overline{x}\overline{y}\overline{z} + \underline{x}\overline{y}\overline{z}$$

$$w = \overline{y}\overline{z} x + \frac{\theta_3}{\overline{y} - \underline{y}} y + \overline{x}\overline{y} z + \left(-\frac{\theta_3 \underline{y}}{\overline{y} - \underline{y}} - \overline{x}\overline{y}\underline{z} - \underline{x}\overline{y}\overline{z} + \underline{x}\underline{y}\underline{z} \right),$$

where

$$\theta_3 = \overline{x}\overline{y}\underline{z} - \underline{x}\underline{y}\underline{z} - \overline{x}\overline{y}\overline{z} + \underline{x}\overline{y}\overline{z}.$$

If neither of the above sets of conditions apply, mapping $\{x_1, x_2, x_3\}$ onto $\{x, y, z\}$ in such a way that the following relation applies,

$$\underline{x}\underline{y}\underline{z} + \overline{x}\overline{y}\underline{z} + \overline{x}\underline{y}\overline{z} \geq \underline{x}\overline{y}\overline{z} + 2\overline{x}\underline{y}\underline{z},$$

the linear equalities defining the facets of $\underline{\mathcal{C}}_3(x)$ are:

$$w = \overline{y}\underline{z} x + \underline{x}\underline{z} y + \underline{x}\overline{y} z - 2\underline{x}\overline{y}\underline{z}$$

$$w = \underline{y}\overline{z} x + \underline{x}\overline{z} y + \underline{x}\underline{y} z - 2\underline{x}\underline{y}\overline{z}$$

$$w = \overline{y}\overline{z} x + \overline{x}\overline{z} y + \overline{x}\overline{y} z - 2\overline{x}\overline{y}\overline{z}$$

$$w = \underline{y}\underline{z} x + \frac{\theta_1}{\underline{y} - \overline{y}} y + \overline{x}\underline{y} z + \left(-\frac{\theta_1 \overline{y}}{\underline{y} - \overline{y}} - \underline{x}\underline{y}\underline{z} - \overline{x}\underline{y}\overline{z} + \underline{x}\overline{y}\overline{z} \right),$$

where

$$\theta_1 = \underline{x}\underline{y}\underline{z} - \underline{x}\overline{y}\overline{z} - \overline{x}\underline{y}\underline{z} + \overline{x}\underline{y}\overline{z}$$

$$\begin{aligned} w = {} & \frac{\theta_2}{\overline{x} - \underline{x}} x + \overline{x}\underline{z} y + \overline{x}\underline{y} z \\ & + \left(-\frac{\theta_2 \underline{x}}{\overline{x} - \underline{x}} - \overline{x}\overline{y}\underline{z} - \overline{x}\underline{y}\overline{z} + \underline{x}\overline{y}\overline{z} \right), \end{aligned}$$

where

$$\theta_2 = \overline{x}\overline{y}\underline{z} - \underline{x}\overline{y}\overline{z} - \overline{x}\underline{y}\underline{z} + \overline{x}\underline{y}\overline{z}$$

$$\begin{aligned} w = {} & \underline{y}\underline{z} x + \overline{x}\underline{z} y + \frac{\theta_3}{\underline{z} - \overline{z}} z \\ & + \left(-\frac{\theta_3 \overline{z}}{\underline{z} - \overline{z}} - \underline{x}\underline{y}\underline{z} - \overline{x}\overline{y}\underline{z} + \underline{x}\overline{y}\overline{z} \right), \end{aligned}$$

where

$$\theta_3 = \underline{x}\underline{y}\underline{z} - \underline{x}\overline{y}\overline{z} - \overline{x}\underline{y}\underline{z} + \overline{x}\overline{y}\underline{z}.$$

3.4. CASE 4: $\underline{x} \geq 0,\ \underline{y} \leq 0,\ \overline{z} \leq 0,\ \overline{y} \geq 0$

Mapping $\{x_1, x_2, x_3\}$ onto $\{x, y, z\}$ the linear equalities defining the facets of $\underline{\mathcal{C}}_3(x)$ are:

$$w = \underline{y}\underline{z} x + \overline{x}\underline{z} y + \overline{x}\underline{y} z - 2\overline{x}\underline{y}\underline{z}$$

$$w = \overline{y}\underline{z} x + \underline{x}\overline{z} y + \underline{x}\overline{y} z - \underline{x}\overline{y}\underline{z} - \underline{x}\overline{y}\overline{z}$$

$$w = \overline{y}\underline{z} x + \underline{x}\underline{z} y + \underline{x}\underline{y} z - \underline{x}\overline{y}\underline{z} - \underline{x}\underline{y}\underline{z}$$

$$w = \underline{y}\overline{z} x + \overline{x}\overline{z} y + \overline{x}\overline{y} z - \overline{x}\underline{y}\overline{z} - \overline{x}\overline{y}\overline{z}$$

$$w = \overline{y}\overline{z} x + \underline{x}\overline{z} y + \overline{x}\overline{y} z - \underline{x}\overline{y}\overline{z} - \overline{x}\overline{y}\overline{z}$$

$$\begin{aligned} w = {} & \underline{y}\overline{z} x + \frac{\theta}{\underline{y} - \overline{y}} y + \underline{x}\underline{y} z \\ & + \left(-\frac{\theta \overline{y}}{\underline{y} - \overline{y}} - \overline{x}\underline{y}\overline{z} - \underline{x}\underline{y}\underline{z} + \overline{x}\overline{y}\underline{z} \right), \end{aligned}$$

where

$$\theta = \overline{x}\underline{y}\overline{z} - \overline{x}\overline{y}\underline{z} - \underline{x}\underline{y}\overline{z} + \underline{x}\underline{y}\underline{z}.$$

3.5. CASE 5: $\underline{x} \leq 0,\ \underline{y} \leq 0,\ \overline{z} \leq 0,\ \overline{x} \geq 0,\ \overline{y} \geq 0$

Mapping $\{x_1, x_2, x_3\}$ onto $\{x, y, z\}$, if the following condition holds:

$$\underline{x}\underline{y}\underline{z} + \overline{x}\overline{y}\overline{z} \leq \overline{x}\overline{y}\underline{z} + \underline{x}\underline{y}\overline{z}$$

the linear equalities defining the facets of $\underline{\mathcal{C}}_3(x)$ are:

$$w = \overline{y}\underline{z}x + \underline{xz}y + \underline{x}\overline{y}z - 2\underline{x}\overline{y}\underline{z}$$

$$w = \underline{yz}x + \overline{x}\underline{z}y + \overline{x}\underline{y}z - 2\overline{x}\underline{yz}$$

$$w = \overline{yz}x + \underline{x}\overline{z}y + \underline{xy}z - \underline{x}\overline{yz} - \underline{xy}\overline{z}$$

$$w = \underline{y}\overline{z}x + \overline{xz}y + \underline{xy}z - \overline{x}\underline{y}\overline{z} - \underline{xy}\overline{z}$$

$$w = \frac{\theta_1}{\overline{x} - \underline{x}}x + \overline{xz}y + \overline{xy}z + \left(-\frac{\theta_1\underline{x}}{\overline{x} - \underline{x}} - \overline{xy}\underline{z} - \overline{x}\underline{y}\overline{z} + \underline{xyz}\right),$$

where

$$\theta_1 = \overline{xy}\underline{z} - \underline{xyz} - \overline{xyz} + \overline{x}\underline{y}\overline{z}$$

$$w = \overline{yz}x + \frac{\theta_2}{\overline{y} - \underline{y}}y + \overline{xy}z + \left(-\frac{\theta_2\underline{y}}{\overline{y} - \underline{y}} - \overline{xy}\underline{z} - \underline{x}\overline{yz} + \underline{xyz}\right),$$

where

$$\theta_2 = \overline{xy}\underline{z} - \underline{xyz} - \overline{xyz} + \underline{x}\overline{yz}.$$

Otherwise, the linear equalities defining the facets of $\underline{\mathcal{C}}_3(x)$ are:

$$w = \overline{y}\underline{z}x + \underline{xz}y + \underline{x}\overline{y}z - 2\underline{x}\overline{y}\underline{z}$$

$$w = \underline{yz}x + \overline{x}\underline{z}y + \overline{x}\underline{y}z - 2\overline{x}\underline{yz}$$

$$w = \underline{y}\overline{z}x + \overline{xz}y + \overline{xy}z - \overline{x}\underline{y}\overline{z} - \overline{xyz}$$

$$w = \overline{yz}x + \underline{x}\overline{z}y + \overline{xy}z - \underline{x}\overline{yz} - \overline{xyz}$$

$$w = \frac{\theta_1}{\underline{x} - \overline{x}}x + \underline{x}\overline{z}y + \underline{xy}z + (-\frac{\theta_1\overline{x}}{\underline{x} - \overline{x}} - \underline{xyz} - \underline{x}\overline{yz} + \overline{xy}\underline{z}),$$

where

$$\theta_1 = \underline{xyz} - \overline{xy}\underline{z} - \underline{xy}\overline{z} + \underline{x}\overline{yz}$$

$$w = \underline{y}\overline{z}x + \frac{\theta_2}{\underline{y} - \overline{y}}y + \underline{xy}z + \left(-\frac{\theta_2\overline{y}}{\underline{y} - \overline{y}} - \underline{xyz} - \overline{x}\underline{y}\overline{z} + \overline{xy}\underline{z}\right),$$

where

$$\theta_2 = \underline{x}\underline{y}\underline{z} - \overline{x}\overline{y}\underline{z} - \underline{x}\underline{y}\overline{z} + \overline{x}\underline{y}\overline{z}.$$

3.6. CASE 6: $\underline{x} \le 0,\ \overline{x} \ge 0,\ \overline{y} \le 0,\ \overline{z} \le 0$

Mapping $\{x_1, x_2, x_3\}$ onto $\{x, y, z\}$ the linear equalities defining the facets of $\underline{\mathcal{C}}_3(x)$ are:

$$w = \underline{y}\underline{z}x + \overline{x}\underline{z}y + \overline{x}\underline{y}z - 2\overline{x}\underline{y}\underline{z}$$

$$w = \overline{y}\overline{z}x + \underline{x}\underline{z}y + \underline{x}\overline{y}z - \underline{x}\overline{y}\overline{z} - \underline{x}\overline{y}\underline{z}$$

$$w = \overline{y}\overline{z}x + \underline{x}\overline{z}y + \underline{x}\underline{y}z - \underline{x}\overline{y}\overline{z} - \underline{x}\underline{y}\overline{z}$$

$$w = \underline{y}\overline{z}x + \overline{x}\overline{z}y + \underline{x}\underline{y}z - \overline{x}\underline{y}\overline{z} - \underline{x}\underline{y}\overline{z}$$

$$w = \overline{y}\underline{z}x + \underline{x}\underline{z}y + \overline{x}\overline{y}z - \underline{x}\overline{y}\underline{z} - \overline{x}\overline{y}\underline{z}$$

$$w = \frac{\theta}{\overline{x} - \underline{x}}x + \overline{x}\overline{z}y + \overline{x}\overline{y}z + \left(-\frac{\theta\underline{x}}{\overline{x} - \underline{x}} - \overline{x}\overline{y}\underline{z} - \overline{x}\underline{y}\overline{z} + \underline{x}\underline{y}\underline{z}\right),$$

where

$$\theta = \overline{x}\overline{y}\underline{z} - \underline{x}\underline{y}\underline{z} - \overline{x}\overline{y}\overline{z} + \overline{x}\underline{y}\overline{z}.$$

3.7. ILLUSTRATION 1

To construct the lower bounding facets of $\underline{\mathcal{C}}_3(x)$ where $\mathbf{x} = [-1, 2] \times [-5, -2] \times [-3, 5]$ we first observe that there are three negative lower bounds and one negative upper bound, so *Case 5* applies. We substitute $z \leftarrow x_2$, as x_2 is the only variable with both negative lower and upper bounds. If, arbitrarily, we substitute $x \leftarrow x_1$ and $y \leftarrow x_3$ the condition,

$$\underline{x}\underline{y}\underline{z} + \overline{x}\overline{y}\overline{z} \le \overline{x}\overline{y}\underline{z} + \underline{x}\underline{y}\overline{z}.$$

translates into the form,

$$\underline{x}_1\underline{x}_2\underline{x}_3 + \overline{x}_1\overline{x}_2\overline{x}_3 \le \overline{x}_1\underline{x}_2\overline{x}_3 + \underline{x}_1\overline{x}_2\underline{x}_3$$
$$(-1)(-5)(-3) + (2)(-2)(5) \le (2)(-5)(5) + (-1)(-2)(-3)$$
$$-35 \le -56,$$

so the condition does not hold. Therefore we use the second set of equations for *Case 5* which, on substituting $x \leftarrow x_1$, $z \leftarrow x_2$, $y \leftarrow x_3$ is as follows:

$$w = \overline{x}_3\underline{x}_2x_1 + \underline{x}_1\underline{x}_2x_3 + \underline{x}_1\overline{x}_3x_2 - 2\underline{x}_1\overline{x}_3\underline{x}_2$$

$$w = \underline{x}_3\underline{x}_2x_1 + \overline{x}_1\underline{x}_2x_3 + \overline{x}_1\underline{x}_3x_2 - 2\overline{x}_1\underline{x}_3\underline{x}_2$$

$$w = \underline{x}_3\overline{x}_2x_1 + \overline{x}_1\overline{x}_2x_3 + \overline{x}_1\overline{x}_3x_2 - \overline{x}_1\underline{x}_3\overline{x}_2 - \overline{x}_1\overline{x}_3\overline{x}_2$$

$$w = \overline{x}_3\overline{x}_2x_1 + \underline{x}_1\overline{x}_2x_3 + \overline{x}_1\overline{x}_3x_2 - \underline{x}_1\overline{x}_3\overline{x}_2 - \overline{x}_1\overline{x}_3\overline{x}_2$$

$$w = \frac{\theta_1}{\underline{x}_1 - \overline{x}_1} x_1 + \underline{x}_1 \overline{x}_2 x_3 + \underline{x}_1 \underline{x}_3 x_2 + \left(-\frac{\theta_1 \overline{x}_1}{\underline{x}_1 - \overline{x}_1} - \underline{x}_1 \underline{x}_3 \underline{x}_2 - \underline{x}_1 \overline{x}_3 \overline{x}_2 + \overline{x}_1 \overline{x}_3 \underline{x}_2 \right),$$

where

$$\theta_1 = \underline{x}_1 \underline{x}_3 \underline{x}_2 - \overline{x}_1 \overline{x}_3 \underline{x}_2 - \underline{x}_1 \underline{x}_3 \overline{x}_2 + \underline{x}_1 \overline{x}_3 \overline{x}_2$$

$$w = \underline{x}_3 \overline{x}_2 x_1 + \frac{\theta_2}{\underline{x}_3 - \overline{x}_3} x_3 + \underline{x}_1 \underline{x}_3 x_2 + \left(-\frac{\theta_2 \overline{x}_3}{\underline{x}_3 - \overline{x}_3} - \underline{x}_1 \underline{x}_3 \underline{x}_2 - \overline{x}_1 \underline{x}_3 \overline{x}_2 + \overline{x}_1 \overline{x}_3 \underline{x}_2 \right),$$

where

$$\theta_2 = \underline{x}_1 \underline{x}_3 \underline{x}_2 - \overline{x}_1 \overline{x}_3 \underline{x}_2 - \underline{x}_1 \underline{x}_3 \overline{x}_2 + \overline{x}_1 \underline{x}_3 \overline{x}_2.$$

Substituting values we get,

$$w = -25x_1 + 5x_3 - 5x_2 - 50,$$
$$w = 15x_1 - 10x_3 - 6x_2 - 60,$$
$$w = 6x_1 - 4x_3 + 10x_2 + 8,$$
$$w = -10x_1 + 2x_3 + 10x_2 + 10,$$
$$w = -17x_1 + 2x_3 + 3x_2 - 11,$$
$$w = 6x_1 - 6.625x_3 + 3x_2 - 13.875.$$

4. Main Results: Facets of the Concave Envelope

The description of the nonvertical facets of $\overline{\mathcal{C}}_3(\mathbf{x})$ for the six cases is presented in this section.

4.1. CASE 1: $\underline{x} \geq 0$, $\underline{y} \geq 0$, $\underline{z} \leq 0$, $\overline{z} \geq 0$

Mapping $\{x_1, x_2, x_3\}$ onto $\{x, y, z\}$ the linear equalities defining the facets of $\overline{\mathcal{C}}_3(x)$ are:

$$w = \overline{y}\underline{z}x + \overline{x}\underline{z}y + \overline{x}\overline{y}z - 2\overline{x}\overline{y}\underline{z}$$
$$w = \underline{y}\underline{z}x + \overline{x}\overline{z}y + \overline{x}\underline{y}z - \overline{x}\underline{y}\overline{z} - \overline{x}\underline{y}\underline{z}$$
$$w = \overline{y}\overline{z}x + \underline{x}\overline{z}y + \underline{x}\underline{y}z - \underline{x}\overline{y}\overline{z} - \underline{x}\underline{y}\overline{z}$$
$$w = \overline{y}\overline{z}x + \underline{x}\underline{z}y + \underline{x}\overline{y}z - \underline{x}\overline{y}\overline{z} - \underline{x}\overline{y}\underline{z}$$
$$w = \underline{y}\overline{z}x + \overline{x}\overline{z}y + \underline{x}\underline{y}z - \overline{x}\underline{y}\overline{z} - \underline{x}\underline{y}\overline{z}$$

$$
\begin{aligned}
w =& \underline{y}\underline{z}x + \underline{x}\underline{z}y + \frac{\theta}{\underline{z}-\overline{z}}z \\
&+ \left(-\frac{\theta\overline{z}}{\underline{z}-\overline{z}} - \overline{x}\underline{y}\underline{z} - \underline{x}\overline{y}\underline{z} + \overline{x}\overline{y}\overline{z} \right),
\end{aligned}
$$

where

$$\theta = \overline{x}\underline{y}\underline{z} - \overline{x}\overline{y}\overline{z} - \underline{x}\underline{y}\underline{z} + \underline{x}\overline{y}\underline{z}.$$

4.2. CASE 2: $\underline{x} \geq 0,\ \underline{y} \leq 0,\ \underline{z} \leq 0,\ \overline{y} \geq 0,\ \overline{z} \geq 0$

Mapping $\{x_1, x_2, x_3\}$ onto $\{x, y, z\}$, if following relation applies:

$$\underline{x}\underline{y}\underline{z} + \overline{x}\overline{y}\overline{z} \geq \overline{x}\underline{y}\underline{z} + \underline{x}\overline{y}\overline{z},$$

the linear equalities defining the facets of $\overline{\mathcal{C}}_3(x)$ are:

$$
\begin{aligned}
w =& \underline{y}\overline{z}x + \overline{x}\overline{z}y + \overline{x}\underline{y}z - 2\overline{x}\underline{y}\overline{z} \\
w =& \overline{y}\underline{z}x + \overline{x}\underline{z}y + \overline{x}\overline{y}z - 2\overline{x}\overline{y}\underline{z} \\
w =& \overline{y}\overline{z}x + \underline{x}\overline{z}y + \underline{x}\underline{y}z - \underline{x}\overline{y}\overline{z} - \underline{x}\underline{y}\overline{z} \\
w =& \overline{y}\overline{z}x + \underline{x}\underline{z}y + \underline{x}\overline{y}z - \underline{x}\overline{y}\overline{z} - \underline{x}\overline{y}\underline{z} \\
w =& \underline{y}\underline{z}x + \underline{x}\underline{z}y + \frac{\theta_1}{\underline{z}-\overline{z}}z \\
&+ \left(-\frac{\theta_1\overline{z}}{\underline{z}-\overline{z}} - \overline{x}\underline{y}\underline{z} - \underline{x}\overline{y}\underline{z} + \overline{x}\overline{y}\overline{z} \right),
\end{aligned}
$$

where

$$
\begin{aligned}
\theta_1 =& \overline{x}\underline{y}\underline{z} - \overline{x}\overline{y}\overline{z} - \underline{x}\underline{y}\underline{z} + \underline{x}\overline{y}\underline{z} \\
w =& \underline{y}\underline{z}x + \frac{\theta_2}{\underline{y}-\overline{y}}y + \underline{x}\underline{y}z \\
&+ \left(-\frac{\theta_2\overline{y}}{\underline{y}-\overline{y}} - \overline{x}\underline{y}\underline{z} - \underline{x}\underline{y}\overline{z} + \overline{x}\overline{y}\overline{z} \right),
\end{aligned}
$$

where

$$\theta_2 = \overline{x}\underline{y}\underline{z} - \overline{x}\overline{y}\overline{z} - \underline{x}\underline{y}\underline{z} + \underline{x}\underline{y}\overline{z}.$$

otherwise the linear equalities defining the facets of $\overline{\mathcal{C}}_3(x)$ are:

$$
\begin{aligned}
w =& \underline{y}\overline{z}x + \overline{x}\overline{z}y + \overline{x}\underline{y}z - 2\overline{x}\underline{y}\overline{z} \\
w =& \overline{y}\underline{z}x + \overline{x}\underline{z}y + \overline{x}\overline{y}z - 2\overline{x}\overline{y}\underline{z} \\
w =& \underline{y}\underline{z}x + \underline{x}\underline{z}y + \underline{x}\overline{y}z - \underline{x}\underline{y}\underline{z} - \underline{x}\overline{y}\underline{z} \\
w =& \underline{y}\underline{z}x + \underline{x}\overline{z}y + \underline{x}\underline{y}z - \underline{x}\underline{y}\underline{z} - \underline{x}\underline{y}\overline{z}
\end{aligned}
$$

$$w = \bar{y}\bar{z}x + \underline{x}\bar{z}y + \frac{\theta_1}{\bar{z} - \underline{z}}z + \left(-\frac{\theta_1\underline{z}}{\bar{z} - \underline{z}} - \bar{x}\bar{y}\bar{z} - \underline{x}\underline{y}\bar{z} + \bar{x}\underline{y}\underline{z}\right),$$

where

$$\theta_1 = \bar{x}\bar{y}\bar{z} - \bar{x}\underline{y}\underline{z} - \underline{x}\bar{y}\bar{z} + \underline{x}\underline{y}\bar{z}$$

$$w = \bar{y}\bar{z}x + \frac{\theta_2}{\bar{y} - \underline{y}}y + \underline{x}\bar{y}z + \left(-\frac{\theta_2\underline{y}}{\bar{y} - \underline{y}} - \bar{x}\bar{y}\bar{z} - \underline{x}\bar{y}\underline{z} + \bar{x}\underline{y}\underline{z}\right),$$

where

$$\theta_2 = \bar{x}\bar{y}\bar{z} - \bar{x}\underline{y}\underline{z} - \underline{x}\bar{y}\bar{z} + \underline{x}\bar{y}\underline{z}.$$

4.3. CASE 3: $\underline{x} \leq 0,\ \underline{y} \leq 0,\ \underline{z} \leq 0,\ \bar{x} \geq 0,\ \bar{y} \geq 0,\ \bar{z} \geq 0$

Map $\{x_1, x_2, x_3\}$ onto $\{x, y, z\}$. If the following conditions apply:

$$\bar{x}\underline{y}\underline{z} + \underline{x}\bar{y}\underline{z} + \underline{x}\underline{y}\bar{z} \geq \bar{x}\bar{y}\bar{z} + 2\underline{x}\underline{y}\underline{z},$$

$$\underline{x}\bar{y}\underline{z} + \underline{x}\underline{y}\bar{z} + \bar{x}\bar{y}\bar{z} \geq \bar{x}\underline{y}\underline{z} + 2\underline{x}\bar{y}\bar{z},$$

$$\bar{x}\underline{y}\underline{z} + \underline{x}\underline{y}\bar{z} + \bar{x}\bar{y}\bar{z} \geq \underline{x}\bar{y}\underline{z} + 2\bar{x}\underline{y}\bar{z},$$

$$\bar{x}\underline{y}\underline{z} + \underline{x}\bar{y}\underline{z} + \bar{x}\bar{y}\bar{z} \geq \underline{x}\underline{y}\bar{z} + 2\bar{x}\bar{y}\underline{z},$$

the linear equalities defining the facets of $\overline{C}_3(x)$ are:

$$w = \underline{y}\underline{z}x + \underline{x}\underline{z}y + \underline{x}\underline{y}z - 2\underline{x}\underline{y}\underline{z} \tag{4.21}$$

$$w = \bar{y}\underline{z}x + \bar{x}\underline{z}y + \bar{x}\bar{y}z - 2\bar{x}\bar{y}\underline{z} \tag{4.22}$$

$$w = \underline{y}\bar{z}x + \bar{x}\bar{z}y + \bar{x}\underline{y}z - 2\bar{x}\underline{y}\bar{z} \tag{4.23}$$

$$w = \bar{y}\bar{z}x + \underline{x}\bar{z}y + \underline{x}\bar{y}z - 2\underline{x}\bar{y}\bar{z} \tag{4.24}$$

$$w = \theta_x x + \theta_y y + \theta_z z + \theta_c, \tag{4.25}$$

where

$$\theta_x = \frac{1}{2}(\bar{x}\underline{y}\underline{z} + \bar{x}\bar{y}\bar{z} - \underline{x}\underline{y}\bar{z} - \underline{x}\bar{y}\underline{z})/(\bar{x} - \underline{x})$$

$$\theta_y = \frac{1}{2}(\underline{x}\bar{y}\underline{z} + \bar{x}\bar{y}\bar{z} - \underline{x}\underline{y}\bar{z} - \bar{x}\underline{y}\underline{z})/(\bar{y} - \underline{y})$$

$$\theta_z = \frac{1}{2}(\underline{x}\underline{y}\overline{z} + \overline{xyz} - \underline{x}\overline{y}\underline{z} - \overline{x}\underline{y}\underline{z})/(\overline{z} - \underline{z})$$
$$\theta_c = \overline{xyz} - \theta_x\overline{x} - \theta_y\overline{y} - \theta_z\overline{z}.$$

Otherwise if the following conditions apply:

$$\overline{x}\underline{y}\underline{z} + \underline{x}\overline{y}\underline{z} + \underline{x}\underline{y}\overline{z} \leq \overline{xyz} + 2\underline{x}\underline{y}\underline{z},$$

the linear equalities defining the facets of $\overline{C}_3(x)$ are:

$$w = \overline{y}\underline{z}x + \overline{x}\underline{z}y + \overline{xy}z - 2\overline{xy}\underline{z}$$
$$w = \underline{y}\overline{z}x + \overline{xz}y + \overline{x}\underline{y}z - 2\overline{x}\underline{y}\overline{z}$$
$$w = \overline{yz}x + \underline{x}\overline{z}y + \underline{x}\overline{y}z - 2\underline{x}\overline{yz}$$
$$w = \underline{y}\underline{z}x + \underline{x}\underline{z}y + \frac{\theta_1}{\underline{z} - \overline{z}}z + \left(-\frac{\theta_1\overline{z}}{\underline{z} - \overline{z}} - \overline{x}\underline{y}\underline{z} - \underline{x}\overline{y}\underline{z} + \overline{xyz}\right),$$

where

$$\theta_1 = \overline{x}\underline{y}\underline{z} - \overline{xyz} - \underline{x}\underline{y}\underline{z} + \underline{x}\overline{y}\underline{z}$$
$$w = \frac{\theta_2}{\underline{x} - \overline{x}}x + \underline{x}\underline{z}y + \underline{x}\underline{y}z + \left(-\frac{\theta_2\overline{x}}{\underline{x} - \overline{x}} - \underline{x}\underline{y}\overline{z} - \underline{x}\overline{y}\underline{z} + \overline{xyz}\right),$$

where

$$\theta_2 = \underline{x}\underline{y}\overline{z} - \overline{xyz} - \underline{x}\underline{y}\underline{z} + \underline{x}\overline{y}\underline{z}$$
$$w = \underline{y}\underline{z}x + \frac{\theta_3}{\underline{y} - \overline{y}}y + \underline{x}\underline{y}z + \left(-\frac{\theta_3\overline{y}}{\underline{y} - \overline{y}} - \underline{x}\underline{y}\overline{z} - \overline{x}\underline{y}\underline{z} + \overline{xyz}\right),$$

where

$$\theta_3 = \underline{x}\underline{y}\overline{z} - \overline{xyz} - \underline{x}\underline{y}\underline{z} + \overline{x}\underline{y}\underline{z}.$$

If neither of the above sets of conditions apply, mapping $\{x_1, x_2, x_3\}$ onto $\{x, y, z\}$ in such a way that the following relation applies,

$$\underline{x}\overline{y}\underline{z} + \underline{x}\underline{y}\overline{z} + \overline{xyz} \leq \overline{x}\underline{y}\underline{z} + 2\underline{x}\overline{yz},$$

the linear equalities defining the facets of $\overline{\mathcal{C}}_3(x)$ are:

$$
\begin{aligned}
w &= \underline{y}\underline{z}x + \underline{x}\underline{z}y + \underline{x}\underline{y}z - 2\underline{x}\underline{y}\underline{z}\\
w &= \overline{y}\underline{z}x + \overline{x}\underline{z}y + \overline{x}\overline{y}z - 2\overline{x}\overline{y}\underline{z}\\
w &= \underline{y}\overline{z}x + \overline{x}\overline{z}y + \overline{x}\underline{y}z - 2\overline{x}\underline{y}\overline{z}\\
w &= \frac{\theta_1}{\underline{x}-\overline{x}}x + \underline{x}\overline{z}y + \underline{x}\overline{y}z\\
&\quad + \left(-\frac{\theta_1\overline{x}}{\underline{x}-\overline{x}} - \underline{x}\underline{y}\overline{z} - \underline{x}\overline{y}\underline{z} + \overline{x}\underline{y}\underline{z}\right),
\end{aligned}
$$

where

$$
\begin{aligned}
\theta_1 &= \underline{x}\underline{y}\overline{z} - \overline{x}\underline{y}\underline{z} - \underline{x}\overline{y}\overline{z} + \underline{x}\overline{y}\underline{z}\\
w &= \overline{y}\overline{z}x + \frac{\theta_2}{\overline{y}-\underline{y}}y + \underline{x}\overline{y}z\\
&\quad + \left(-\frac{\theta_2\underline{y}}{\overline{y}-\underline{y}} - \overline{x}\overline{y}\overline{z} - \underline{x}\overline{y}\underline{z} + \overline{x}\underline{y}\underline{z}\right),
\end{aligned}
$$

where

$$
\begin{aligned}
\theta_2 &= \overline{x}\overline{y}\overline{z} - \overline{x}\underline{y}\underline{z} - \underline{x}\overline{y}\overline{z} + \underline{x}\overline{y}\underline{z}\\
w &= \overline{y}\overline{z}x + \underline{x}\overline{z}y + \frac{\theta_3}{\overline{z}-\underline{z}}z\\
&\quad + \left(-\frac{\theta_3\underline{z}}{\overline{z}-\underline{z}} - \underline{x}\underline{y}\overline{z} - \overline{x}\overline{y}\overline{z} + \overline{x}\underline{y}\underline{z}\right),
\end{aligned}
$$

where

$$\theta_3 = \underline{x}\underline{y}\overline{z} - \overline{x}\underline{y}\underline{z} - \underline{x}\overline{y}\overline{z} + \overline{x}\overline{y}\overline{z}.$$

4.4. CASE 4: $\underline{x} \geq 0,\ \underline{y} \leq 0,\ \overline{z} \leq 0,\ \overline{y} \geq 0$

Mapping $\{x_1, x_2, x_3\}$ onto $\{x, y, z\}$ the linear equalities defining the facets of $\overline{\mathcal{C}}_3(x)$ are:

$$
\begin{aligned}
w &= \overline{y}\underline{z}x + \overline{x}\underline{z}y + \overline{x}\overline{y}z - 2\overline{x}\overline{y}\underline{z}\\
w &= \overline{y}\overline{z}x + \overline{x}\overline{z}y + \overline{x}\underline{y}z - \overline{x}\overline{y}\overline{z} - \overline{x}\underline{y}\overline{z}\\
w &= \underline{y}\underline{z}x + \underline{x}\underline{z}y + \underline{x}\overline{y}z - \underline{x}\overline{y}\underline{z} - \underline{x}\underline{y}\underline{z}\\
w &= \underline{y}\underline{z}x + \underline{x}\overline{z}y + \underline{x}\underline{y}z - \underline{x}\underline{y}\overline{z} - \underline{x}\underline{y}\underline{z}\\
w &= \underline{y}\overline{z}x + \underline{x}\overline{z}y + \overline{x}\underline{y}z - \underline{x}\underline{y}\overline{z} - \overline{x}\underline{y}\overline{z}
\end{aligned}
$$

$$w = \overline{y}\overline{z}x + \frac{\theta}{\overline{y} - \underline{y}}y + \underline{x}\overline{y}z + \left(-\frac{\theta\underline{y}}{\overline{y} - \underline{y}} - \underline{x}\overline{y}\underline{z} - \overline{x}\overline{y}\overline{z} + \overline{x}\underline{y}\underline{z}\right),$$

where

$$\theta = \underline{x}\overline{y}\underline{z} - \overline{x}\underline{y}\underline{z} - \underline{x}\overline{y}\overline{z} + \overline{x}\overline{y}\overline{z}.$$

4.5. CASE 5: $\underline{x} \leq 0,\ \underline{y} \leq 0,\ \overline{z} \leq 0,\ \overline{x} \geq 0,\ \overline{y} \geq 0$

Mapping $\{x_1, x_2, x_3\}$ onto $\{x, y, z\}$ in such a way that the following relation holds,

$$\overline{x}\underline{y}\underline{z} + \underline{x}\overline{y}\overline{z} \geq \underline{x}\overline{y}\underline{z} + \overline{x}\underline{y}\overline{z}$$

the linear equalities defining the facets of $\overline{\mathcal{C}}_3(x)$ are:

$$w = \underline{y}\underline{z}x + \underline{x}\underline{z}y + \underline{x}\underline{y}z - 2\underline{x}\underline{y}\underline{z}$$

$$w = \overline{y}\underline{z}x + \overline{x}\underline{z}y + \overline{x}\overline{y}z - 2\overline{x}\overline{y}\underline{z}$$

$$w = \underline{y}\overline{z}x + \underline{x}\overline{z}y + \overline{x}\underline{y}z - \underline{x}\underline{y}\overline{z} - \overline{x}\underline{y}\overline{z}$$

$$w = \overline{y}\overline{z}x + \overline{x}\overline{z}y + \overline{x}\underline{y}z - \overline{x}\overline{y}\overline{z} - \overline{x}\underline{y}\overline{z}$$

$$w = \frac{\theta_1}{\underline{x} - \overline{x}}x + \underline{x}\overline{z}y + \underline{x}\overline{y}z + \left(-\frac{\theta_1\overline{x}}{\underline{x} - \overline{x}} - \underline{x}\overline{y}\underline{z} - \underline{x}\underline{y}\overline{z} + \overline{x}\underline{y}\underline{z}\right),$$

where

$$\theta_1 = \underline{x}\overline{y}\underline{z} - \overline{x}\underline{y}\underline{z} - \underline{x}\overline{y}\overline{z} + \underline{x}\underline{y}\overline{z}$$

$$w = \overline{y}\overline{z}x + \frac{\theta_2}{\overline{y} - \underline{y}}y + \underline{x}\overline{y}z + \left(-\frac{\theta_2\underline{y}}{\overline{y} - \underline{y}} - \overline{x}\overline{y}\overline{z} - \underline{x}\overline{y}\underline{z} + \overline{x}\underline{y}\underline{z}\right),$$

where

$$\theta_2 = \overline{x}\overline{y}\overline{z} - \overline{x}\underline{y}\underline{z} - \underline{x}\overline{y}\overline{z} + \underline{x}\overline{y}\underline{z}.$$

4.6. CASE 6: $\underline{x} \leq 0,\ \overline{x} \geq 0,\ \overline{y} \leq 0,\ \overline{z} \leq 0$

Mapping $\{x_1, x_2, x_3\}$ onto $\{x, y, z\}$the linear equalities defining the facets of $\overline{\mathcal{C}}_3(x)$ are:

$$w = \underline{y}\underline{z}x + \underline{x}\underline{z}y + \underline{x}\underline{y}z - 2\underline{x}\underline{y}\underline{z}$$

$$w = \underline{y}\bar{z}x + \underline{x}\bar{z}y + \bar{x}\underline{y}z - \underline{x}\underline{y}\bar{z} - \bar{x}\underline{y}\bar{z}$$

$$w = \bar{y}\bar{z}x + \bar{x}\bar{z}y + \bar{x}\underline{y}z - \bar{x}\bar{y}\bar{z} - \bar{x}\underline{y}\bar{z}$$

$$w = \bar{y}\bar{z}x + \bar{x}\underline{z}y + \bar{x}\bar{y}z - \bar{x}\bar{y}\bar{z} - \bar{x}\bar{y}\underline{z}$$

$$w = \bar{y}\underline{z}x + \bar{x}\underline{z}y + \underline{x}\bar{y}z - \underline{x}\bar{y}\underline{z} - \bar{x}\bar{y}\underline{z}$$

$$w = \frac{\theta}{\underline{x} - \bar{x}}x + \underline{x}\bar{z}y + \underline{x}\bar{y}z + \left(-\frac{\theta\bar{x}}{\underline{x} - \bar{x}} - \underline{x}\bar{y}\underline{z} - \underline{x}\underline{y}\bar{z} + \bar{x}\underline{y}\underline{z} \right),$$

where

$$\theta = \underline{x}\bar{y}\underline{z} - \bar{x}\underline{y}\underline{z} - \underline{x}\bar{y}\bar{z} + \underline{x}\underline{y}\bar{z}.$$

4.7. ILLUSTRATION 2

This illustration demonstrates the upper bounding facets of $\overline{\mathcal{C}}_3(x)$ where $\mathbf{x} = [-2, 1] \times [-3, 1] \times [-4, 1]$. As the upper bounds are all positive and the lower bounds are all negative Case 3 applies. With the substitutions $x \leftarrow x_1$, $y \leftarrow x_2$ and $z \leftarrow x_3$ the conditions,

$$\bar{x}\underline{y}\underline{z} + \underline{x}\bar{y}\underline{z} + \underline{x}\underline{y}\bar{z} \geq \bar{x}\bar{y}\bar{z} + 2\underline{x}\underline{y}\underline{z},$$

$$\underline{x}\bar{y}\underline{z} + \underline{x}\underline{y}\bar{z} + \bar{x}\bar{y}\bar{z} \geq \bar{x}\underline{y}\underline{z} + 2\underline{x}\bar{y}\bar{z},$$

$$\bar{x}\underline{y}\underline{z} + \underline{x}\underline{y}\bar{z} + \bar{x}\bar{y}\bar{z} \geq \underline{x}\bar{y}\underline{z} + 2\bar{x}\underline{y}\bar{z},$$

$$\bar{x}\underline{y}\underline{z} + \underline{x}\bar{y}\underline{z} + \bar{x}\bar{y}\bar{z} \geq \underline{x}\underline{y}\bar{z} + 2\bar{x}\bar{y}\underline{z},$$

translate into the form,

$$\bar{x}_1\underline{x}_2\underline{x}_3 + \underline{x}_1\bar{x}_2\underline{x}_3 + \underline{x}_1\underline{x}_2\bar{x}_3 \geq \bar{x}_1\bar{x}_2\bar{x}_3 + 2\underline{x}_1\underline{x}_2\underline{x}_3$$
$$26 \geq -47,$$

$$\underline{x}_1\bar{x}_2\underline{x}_3 + \underline{x}_1\underline{x}_2\bar{x}_3 + \bar{x}_1\bar{x}_2\bar{x}_3 \geq \bar{x}_1\underline{x}_2\underline{x}_3 + 2\underline{x}_1\bar{x}_2\bar{x}_3$$
$$15 \geq 8,$$

$$\bar{x}_1\underline{x}_2\underline{x}_3 + \underline{x}_1\underline{x}_2\bar{x}_3 + \bar{x}_1\bar{x}_2\bar{x}_3 \geq \underline{x}_1\bar{x}_2\underline{x}_3 + 2\bar{x}_1\underline{x}_2\bar{x}_3$$
$$19 \geq 2,$$

$$\bar{x}_1\underline{x}_2\underline{x}_3 + \underline{x}_1\bar{x}_2\underline{x}_3 + \bar{x}_1\bar{x}_2\bar{x}_3 \geq \underline{x}_1\underline{x}_2\bar{x}_3 + 2\bar{x}_1\bar{x}_2\underline{x}_3$$
$$21 \geq -2.$$

All of these conditions hold hence Equations (4.21), (4.22), (4.23), (4.24), and (4.25), define the facets of $\overline{\mathcal{C}}_3(x)$:

$$w = \underline{x}_2\underline{x}_3x_1 + \underline{x}_1\underline{x}_3x_2 + \underline{x}_1\underline{x}_2x_3 - 2\underline{x}_1\underline{x}_2\underline{x}_3$$
$$w = \overline{x}_2\underline{x}_3x_1 + \overline{x}_1\underline{x}_3x_2 + \overline{x}_1\overline{x}_2x_3 - 2\overline{x}_1\overline{x}_2\underline{x}_3$$
$$w = \underline{x}_2\overline{x}_3x_1 + \overline{x}_1\overline{x}_3x_2 + \overline{x}_1\underline{x}_2x_3 - 2\overline{x}_1\underline{x}_2\overline{x}_3$$
$$w = \overline{x}_2\overline{x}_3x_1 + \underline{x}_1\overline{x}_3x_2 + \underline{x}_1\overline{x}_2x_3 - 2\underline{x}_1\overline{x}_2\overline{x}_3$$
$$w = \theta_x x_1 + \theta_y x_2 + \theta_z x_3 + \theta_c,$$

where

$$\theta_x = \frac{1}{2}(\overline{x}_1\underline{x}_2\underline{x}_3 + \overline{x}_1\overline{x}_2\overline{x}_3 - \underline{x}_1\underline{x}_2\overline{x}_3 - \underline{x}_1\overline{x}_2\underline{x}_3)/(\overline{x}_1 - \underline{x}_1)$$
$$\theta_y = \frac{1}{2}(\underline{x}_1\overline{x}_2\underline{x}_3 + \overline{x}_1\overline{x}_2\overline{x}_3 - \underline{x}_1\underline{x}_2\overline{x}_3 - \overline{x}_1\underline{x}_2\underline{x}_3)/(\overline{x}_2 - \underline{x}_2)$$
$$\theta_z = \frac{1}{2}(\underline{x}_1\underline{x}_2\overline{x}_3 + \overline{x}_1\overline{x}_2\overline{x}_3 - \underline{x}_1\overline{x}_2\underline{x}_3 - \overline{x}_1\underline{x}_2\underline{x}_3)/(\overline{x}_3 - \underline{x}_3)$$
$$\theta_c = \overline{x}_1\overline{x}_2\overline{x}_3 - \theta_x\overline{x}_1 - \theta_y\overline{x}_2 - \theta_z\overline{x}_3.$$

Substituting values we get,

$$w = 12x_1 + 8x_2 + 6x_3 + 48,$$
$$w = -4x_1 - 4x_2 + 1x_3 + 8,$$
$$w = -3x_1 + 1x_2 - 3x_3 + 6,$$
$$w = 1x_1 - 2x_2 - 2x_3 + 4,$$
$$w = 0.1\dot{6}x_1 - 1.125x_2 - 1.3x_3 + 3.591\dot{6}.$$

5. Proofs for the Facets of the Convex Envelope

In this section we prove that the equalities presented in Sections 3 and 4 define the respective affine hulls of the facets of $\underline{C}_3(\mathbf{x})$ and $\overline{C}_3(\mathbf{x})$. A facet projected onto the domain is a simplex in $\mathfrak{R}^3$. These simplices partition the hyper-rectangle $\mathbf{x}$ into a triangulation. In each case, the set of facets is seen to be complete as the simplices cover $\mathbf{x}$.

The vertices of the hypercube are denoted by: $v_1 := (\underline{x}, \underline{y}, \underline{z})$, $v_2 := (\overline{x}, \underline{y}, \underline{z})$, $v_3 := (\underline{x}, \overline{y}, \underline{z})$, $v_4 := (\overline{x}, \overline{y}, \underline{z})$, $v_5 := (\underline{x}, \underline{y}, \overline{z})$, $v_6 := (\overline{x}, \underline{y}, \overline{z})$, $v_7 := (\underline{x}, \overline{y}, \overline{z})$, $v_8 := (\overline{x}, \overline{y}, \overline{z})$. To facilitate the proofs we introduce the notation $\zeta_1 = \underline{x}\underline{y}\underline{z}$, $\zeta_2 = \overline{x}\underline{y}\underline{z}$, $\zeta_3 = \underline{x}\overline{y}\underline{z}$, $\zeta_4 = \overline{x}\overline{y}\underline{z}$, $\zeta_5 = \underline{x}\underline{y}\overline{z}$, $\zeta_6 = \overline{x}\underline{y}\overline{z}$, $\zeta_7 = \underline{x}\overline{y}\overline{z}$ and $\zeta_8 = \overline{x}\overline{y}\overline{z}$. The following inequalities are used extensively in the proofs:

$$\zeta_1 + \zeta_7 \geq (\leq)\ \zeta_3 + \zeta_5 \quad \text{if } \underline{x} \geq (\leq)\ 0, \tag{5.20}$$

$$\zeta_2 + \zeta_8 \geq (\leq)\ \zeta_4 + \zeta_6 \quad \text{if } \overline{x} \geq (\leq)\ 0, \tag{5.21}$$

$$\zeta_1 + \zeta_6 \geq \ (\leq)\ \zeta_2 + \zeta_5 \quad \text{if } \underline{y} \geq (\leq)\ 0, \tag{5.22}$$

$$\zeta_3 + \zeta_8 \geq \ (\leq)\ \zeta_4 + \zeta_7 \quad \text{if } \overline{y} \geq (\leq)\ 0, \tag{5.23}$$

$$\zeta_1 + \zeta_4 \geq \ (\leq)\ \zeta_2 + \zeta_3 \quad \text{if } \underline{z} \geq (\leq)\ 0, \tag{5.24}$$

$$\zeta_5 + \zeta_8 \geq \ (\leq)\ \zeta_6 + \zeta_7 \quad \text{if } \overline{z} \geq (\leq)\ 0. \tag{5.25}$$

These inequalities are based on the following types of relation:

$$(x - \underline{x})(y - \underline{y}) \geq \ 0 \text{ forall } x \in [\underline{x}, \overline{x}], y \in [\underline{y}, \overline{y}]$$
$$(x - \overline{x})(y - \overline{y}) \geq \ 0 \text{ forall } x \in [\underline{x}, \overline{x}], y \in [\underline{y}, \overline{y}].$$

An auxiliary condition which does not necessarily follow from the signs on the lower and upper bounds is also used in the proofs of cases 1 and 2 for $\underline{\mathcal{C}}_3(\mathbf{x})$,

$$\zeta_4 + \zeta_5 \leq \zeta_3 + \zeta_6. \tag{5.26}$$

The following propositions are statements about two of the facets of $\underline{\mathcal{C}}_3(\mathbf{x})$ for Case 1, where $\underline{x} \geq 0$, $\underline{y} \geq 0$, $\underline{z} \leq 0$, and $\overline{z} \geq 0$. In the first proposition Lemma 2.4. is applied, in the second Lemma 2.3. All the propositions follow a similar format which can be condensed into a tabular form.

PROPOSITION 5.1. *If* $\underline{x} \geq 0, \underline{y} \geq 0$, $\underline{z} \leq 0$, *and* $\overline{z} \geq 0$, *then the following equation defines the affine hull of a facet of* $\underline{\mathcal{C}}_3(x)$:

$$w = \overline{yz}\ x + \overline{xz}\ y + \overline{xy}\ z - 2\overline{xyz}$$

The projected vertices of this facet are:

$$\{v_8,\ v_7,\ v_6,\ v_4\}$$

Proof. Apply Lemma 2.4 with the substitutions: $\chi^A \leftarrow \overline{x}, \psi^A \leftarrow \overline{y}, \omega^A \leftarrow \overline{z}$.
Condition 2.14: $\zeta_4 + \zeta_7 \leq \zeta_8 + \zeta_3$ follows from inequality 5.23.
Condition 2.15: $\zeta_4 + \zeta_6 \leq \zeta_8 + \zeta_2$ follows from inequality 5.21.
Condition 2.16: $\zeta_6 + \zeta_7 \leq \zeta_8 + \zeta_5$ follows from inequality 5.25.
Condition 2.17: $\zeta_4 + \zeta_7 + \zeta_6 \leq 2\zeta_8 + \zeta_1$ follows from inequalities 5.21, 5.22, and 5.25.

PROPOSITION 5.2. *If* $\underline{x} \geq 0$, $\underline{y} \geq 0$, $\underline{z} \leq 0$, *and* $\overline{z} \geq 0$, *then the following equation defines the affine hull of a facet of* $\underline{\mathcal{C}}_3(x)$:

$$w = \overline{y}\underline{z}\ x + \underline{x}\overline{z}\ y + \underline{x}\overline{y}\ z - \underline{x}\overline{y}\underline{z} - \underline{x}\overline{yz}$$

The projected vertices of this facet are:

$$\{v_4,\ v_3,\ v_7,\ v_5\}$$

Proof. Apply Lemma 2.3 with the substitutions: $\chi^A \leftarrow \underline{z}, \psi^A \leftarrow \underline{x}, \omega^A \leftarrow \underline{y}$.
Condition 2.9: $\zeta_5 + \zeta_3 \leq \zeta_7 + \zeta_1$, follows from inequality 5.20.
Condition 2.10: $\zeta_7 + \zeta_4 \leq \zeta_8 + \zeta_3$ follows from inequality 5.23.
Condition 2.11: $\zeta_5 + \zeta_4 \leq \zeta_2 + \zeta_7$ follows from inequalities 5.20 and 5.24.
Condition 2.12: $\zeta_5 + \zeta_4 \leq \zeta_3 + \zeta_6$ follows from inequalities 5.22, and 5.24. □

The remaining facets can be proven in a similar way. Table I summarizes the proofs for the facets of Case 1 and 2 for $\underline{C}_3(\mathbf{x})$. The information in the columns of Table I is as follows:

Facet: The equation number of the facet referred to in the associated table row.

Lem: The number of the lemma from which the facet is derived.
χ^A, ψ^A, ω^A: The symbols substituted for χ^A, ψ^A and ω^A, respectively, in the indicated lemma.

Vertices: The vertices of the simplex defined by the facet projected onto the function domain.

Aux: Auxiliary conditions, other than those implied by the signs of $\underline{x}, \underline{y}, \underline{z}, \overline{x}, \overline{y}$ and $\overline{z}$, which must hold for the facet to be valid.

Cond n: The number before the colon is the equation number of the n^{th} condition in the lemma, the numbers after the colon are the reference numbers of the inequalities from which the n^{th} condition follows.

To clarify this table, consider the first row which refers to the facet defined by Equation (3.19) for Case 1, $\underline{x} \geq 0, \underline{y} \geq 0, \underline{z} \leq 0$, and $\overline{z} \geq 0$. This row summarizes the statement and proof of Proposition 5.1. To prove that this is a facet of $\underline{C}_3(\mathbf{x})$ apply Lemma 2.4, indicated by column "Lem" with the substitutions: $\chi^A \leftarrow \overline{x}, \psi^A \leftarrow \overline{y}, \omega^A \leftarrow \overline{z}$, indicated by columns "$\chi^A$", "$\psi^A$", and "$\omega^A$". The projected vertices of this facet are,

$$\{v_4,\ v_6,\ v_7,\ v_8\ \},$$

indicated in the "Vertices" columns. The simplex defined by these vertices is shown in Figure 1. There are no prerequisite conditions other than the sign conditions, $\underline{x} \geq 0$, $\underline{y} \geq 0$, $\underline{z} \leq 0$, $\overline{x} \geq 0$, $\overline{y} \geq 0$, $\overline{z} \geq 0$, therefore there

Table I. Proofs for Facets of $\mathcal{C}_3$ (**x**), Cases 1 and 2

Facet	Lem	χ^A	ψ^A	ω^A	Vertices				Aux	Cond 1		Cond 2			Cond 3			Cond 4			
Case 1: $\underline{x} \geq 0, \underline{y} \geq 0, \underline{z} \leq 0, \bar{z} \geq 0$																					
3.19	2.4	$\bar{x}$	$\bar{y}$	$\bar{z}$	4	6	7	8		2.14	5.23	2.15:	5.21		2.16	5.25		2.17	5.21	5.22	5.25
3.20	2.3	$\underline{z}$	$\underline{x}$	$\underline{z}$	3	4	5	7		2.9:	5.20	2.10:	5.23		2.11:	5.20	5.24	2.12:	5.22	5.24	
3.21	2.3	$\bar{y}$	$\underline{x}$	$\bar{y}$	1	3	4	5		2.9:	5.20	2.10:	5.24		2.11:	5.21	5.22	2.12:	5.22	5.24	
3.22	2.3	$\bar{z}$	$\bar{x}$	$\bar{y}$	2	4	5	6		2.9:	5.21	2.10:	5.22		2.11:	5.20	5.24	2.12:	5.22	5.24	
3.23	2.3	$\bar{x}$	$\underline{y}$	$\bar{z}$	1	2	4	5		2.9:	5.22	2.10:	5.24		2.11:	5.21	5.22	2.12:	5.20	5.24	
3.24	2.2	$\underline{z}$	$\bar{y}$	$\underline{x}$	4	5	6	7		2.5:	5.25	2.6:	5.22	5.24	2.7:	5.20	5.24	2.8:	5.20	5.22	5.24
Case 2: $\underline{x} \geq 0, \underline{y} \leq 0, \underline{z} \leq 0, \bar{y} \geq 0, \bar{z} \geq 0$																					
3.25	2.4	$\bar{x}$	$\bar{y}$	$\bar{z}$	4	6	7	8		2,14:	5.23	2.15	5.21		2.16:	5.25		2.17:	5.25	5.23	5.20
3.26	2.4	$\bar{x}$	$\underline{z}$	$\underline{y}$	1	2	4	6		2.14:	5.24	2.15:	5.21		2.16:	5.22		2.17:	5.24	5.22	5.20
3.27	2.3	$\underline{z}$	$\underline{x}$	$\bar{z}$	3	4	5	7	5.26	2.9:	5.20	2.10:	5.23		2.11:	5.20	5.24	2.12:	5.26		
3.28	2.3	$\bar{y}$	$\underline{x}$	$\bar{z}$	1	3	4	5	5.26	2.9:	5.20	2.10:	5.24		2.11:	5.20	5.23	2.12:	5.26		
3.29	2.2	$\bar{y}$	$\bar{z}$	$\underline{x}$	1	4	5	6	5.26	2.5:	5.22	2.6:	5.26		2.7:	5.20	5.23	2.8:	5.26	5.20	
3.30	2.2	$\underline{z}$	$\underline{x}$	$\underline{y}$	4	5	6	7	5.26	2.5:	5.25	2.6:	5.20	5.24	2.7:	5.26		2.8:	5.20	5.26	

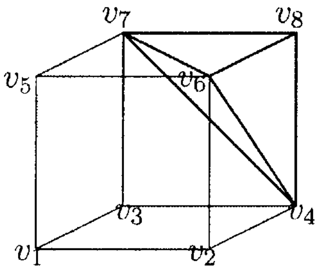

Figure 1. Simplex with vertex points supporting facet in proposition 5.1.

are no "Aux" column entries for this row. Four conditions must hold for Lemma 2.4 to apply. The first is Condition (2.14), indicated before the colon in the "Cond 1" column. Applying the substitutions $\chi^A \leftarrow \overline{x}, \psi^A \leftarrow \overline{y}, \omega^A \leftarrow \overline{z}$, this becomes $\zeta_4 + \zeta_7 \leq \zeta_8 + \zeta_3$ which follows immediately from Inequality (5.23), indicated in the "Cond 1" column after the colon. Similarly, the second condition (2.15), follows from Inequality (5.21), and the third condition (2.16) from (5.25). These are shown in columns "Cond 2" and "Cond 3". The final condition (2.17) follows from Inequalities (5.21), (5.22), and (5.25) as seen in column "Cond 4".

In a similar way Proposition 5.2 can be read from the row of the table for Facet (3.20).

6. Comparison with Other Bounding Schemes

6.1. RECURSIVE ARITHMETIC INTERVALS

Floudas et al. (1989) demonstrated that indefinite quadratic programming problems and polynomial problems may be reduced to bilinear programs through the recursive substitution of products of variables and squares of variables. Hamed (1991) used this "recursive arithmetic interval" (rAI) scheme for generating convex lower bounds for multilinear monomials. Ryoo and Sahinidis (2001) compared this approach with other bounding schemes and showed that the rAI scheme generates the convex and concave envelope for multilinear monomials over the positive orthant. Note that this result is only for the case of the lower bounds being equal to zero. In other words, if we have positive lower bounds the rAI scheme does not provide the convex envelope (Meyer and Floudas, 2003).

To represent the general bound constrained case, $(x, y, z) \in [\underline{x}, \overline{x}] \times [\underline{y}, \overline{y}] \times [\underline{z}, \overline{z}]$, $(\underline{x}, \underline{y}, \underline{z}) \in \Re^3$, four variables are introduced, w_{xyz}, w_{xy}, w_{xz},

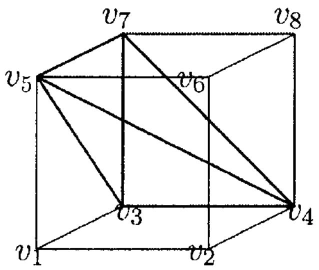

Figure 2. Simplex with vertex points supporting facet in proposition 5.2.

and w_{yz}. The variable w_{xyz}, representing xyz, is bounded from below by the following system of inequalities:

$$
\begin{aligned}
w_{xyz} &\geq \underline{x}w_{yz} + \underline{w}_{yz}x - \underline{x}\underline{w}_{yz} \\
w_{xyz} &\geq \overline{x}w_{yz} + \overline{w}_{yz}x - \overline{x}\overline{w}_{yz}, \\
w_{xyz} &\geq \underline{y}w_{xz} + \underline{w}_{xz}y - \underline{y}\underline{w}_{xz}, \\
w_{xyz} &\geq \overline{y}w_{xz} + \overline{w}_{xz}y - \overline{y}\overline{w}_{xz}, \\
w_{xyz} &\geq \underline{z}w_{xy} + \underline{w}_{xy}z - \underline{z}\underline{w}_{xy}, \\
w_{xyz} &\geq \overline{z}w_{xy} + \overline{w}_{xy}z - \overline{z}\overline{w}_{xy}
\end{aligned}
\tag{6.27}
$$

The bounds, $\underline{w}_{xy}$, $\overline{w}_{xy}$, $\underline{w}_{xz}$, $\underline{w}_{xz}$, $\underline{w}_{yz}$, and $\underline{w}_{yz}$, are determined by,

$$
\begin{aligned}
\underline{w}_{xy} &= \min\{\underline{x}\underline{y}, \underline{x}\overline{y}, \overline{x}\underline{y}, \overline{x}\overline{y}\}, \\
\overline{w}_{xy} &= \max\{\underline{x}\underline{y}, \underline{x}\overline{y}, \overline{x}\underline{y}, \overline{x}\overline{y}\}, \\
\underline{w}_{xz} &= \min\{\underline{x}\underline{z}, \underline{x}\overline{z}, \overline{x}\underline{z}, \overline{x}\overline{z}\}, \\
\overline{w}_{xz} &= \max\{\underline{x}\underline{z}, \underline{x}\overline{z}, \overline{x}\underline{z}, \overline{x}\overline{z}\}, \\
\underline{w}_{yz} &= \min\{\underline{y}\underline{z}, \underline{y}\overline{z}, \overline{y}\underline{z}, \overline{y}\overline{z}\}, \\
\overline{w}_{yz} &= \max\{\underline{y}\underline{z}, \underline{y}\overline{z}, \overline{y}\underline{z}, \overline{y}\overline{z}\}.
\end{aligned}
$$

The variables w_{xy}, w_{xz}, and w_{yz} are substitutes for the respective bilinear terms xy, xz and yz. The bounding schemes of McCormick (1976) and Al-Khayyal and Falk (1983) are used to enclose these terms,

$$
\begin{aligned}
w_{xy} &\geq \underline{y}x + \underline{x}y - \underline{x}\underline{y}, \\
w_{xy} &\geq \overline{y}x + \overline{x}y - \overline{x}\overline{y}, \\
w_{xy} &\leq \overline{y}x + \underline{x}y - \underline{x}\overline{y}, \\
w_{xy} &\leq \underline{y}x + \overline{x}y - \overline{x}\underline{y},
\end{aligned}
$$

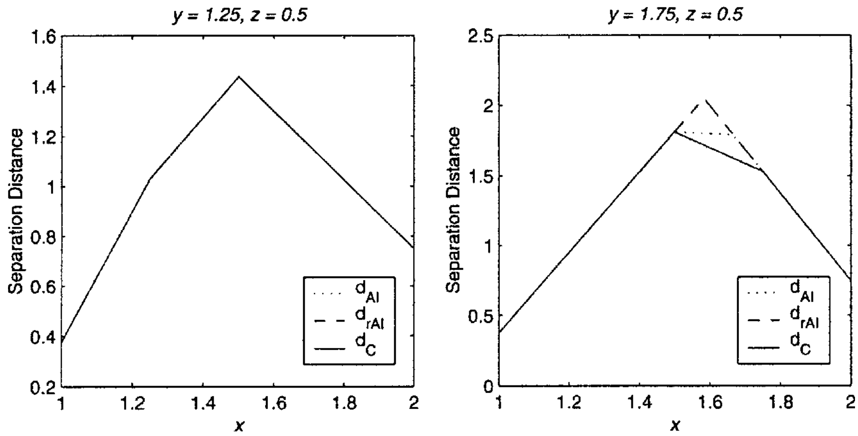

Figure 3. Separation distances for the convex envelope, arithmetic interval and recursive arithmetic interval underestimators $x \in [1,2],\ y \in [1,2],\ z \in [-1,2]$.

$$
\begin{aligned}
w_{xz} &\geq \underline{z}x + \underline{x}z - \underline{x}\underline{z},\\
w_{xz} &\geq \overline{z}x + \overline{x}z - \overline{x}\overline{z},\\
w_{xz} &\leq \overline{z}x + \underline{x}z - \underline{x}\overline{z},\\
w_{xz} &\leq \underline{z}x + \overline{x}z - \overline{x}\underline{z},
\end{aligned}
\tag{6.28}
$$

$$
\begin{aligned}
w_{yz} &\geq \underline{z}y + \underline{y}z - \underline{y}\underline{z},\\
w_{yz} &\geq \overline{z}y + \overline{y}z - \overline{y}\overline{z},\\
w_{yz} &\leq \overline{z}y + \underline{y}z - \underline{y}\overline{z},\\
w_{yz} &\leq \underline{z}y + \overline{y}z - \overline{y}\underline{z}.
\end{aligned}
$$

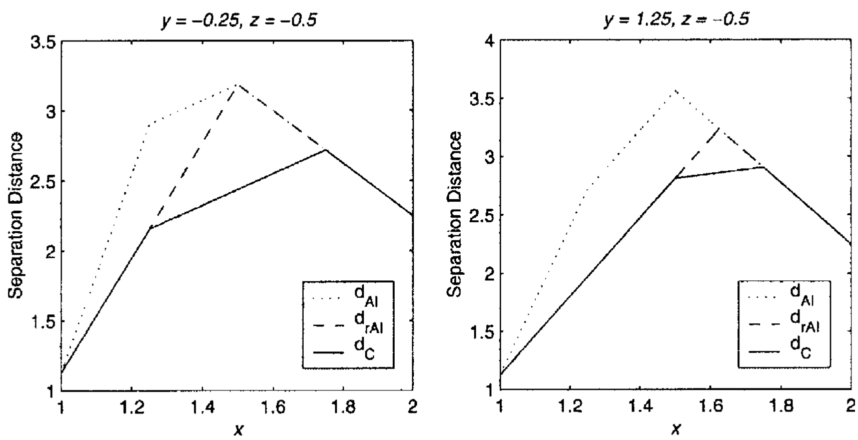

Figure 4. Separation distances for the convex envelope, arithmetic interval and recursive arithmetic interval underestimators $x \in [1,2],\ y \in [-1,2],\ z \in [-2,1]$.

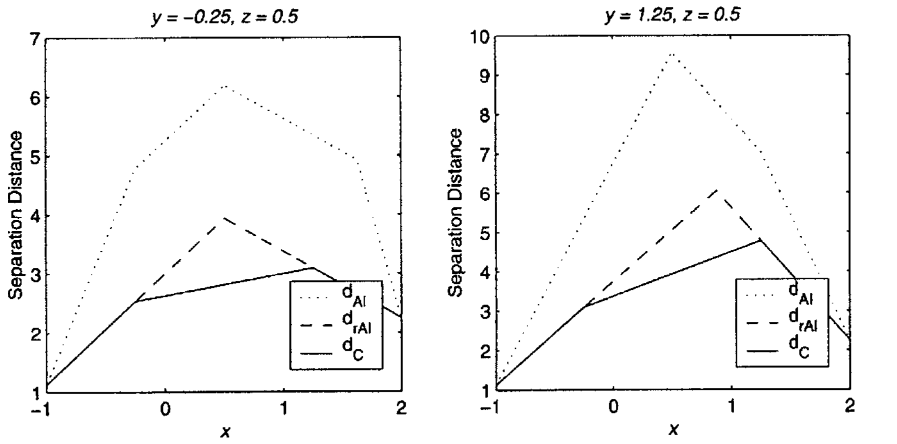

Figure 5. Separation distances for the convex envelope, arithmetic interval and recursive arithmetic interval underestimators $x \in [-1, 2]$, $y \in [-1, 2]$, $z \in [-1, 2]$.

Using this system, a lower bounding function $f_{\mathrm{rAI}}(x, y, z)$ is defined as follows:

$$f_{\mathrm{rAI}}(x, y, z) = \min_{w_{xyz}, w_{xy}, w_{xz}, w_{yz}} w_{xyz} \quad \text{subject to (6.27) and (6.28)}.$$

It should be noted that it is not necessary to introduce all three of the variables, w_{xy}, w_{xz}, and w_{yz} to represent a trilinear term, one is sufficient. The case considered here, where all three are used, is the one that gives the strongest representation using the rAI scheme.

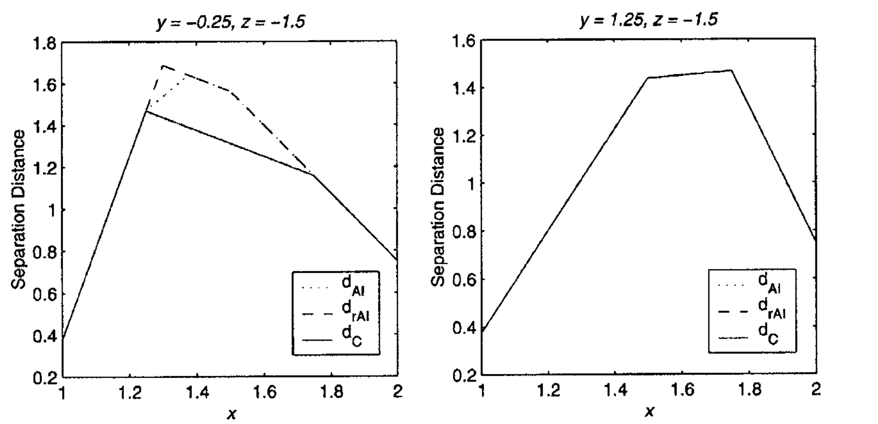

Figure 6. Separation distances for the convex envelope, arithmetic interval and recursive arithmetic interval underestimators $x \in [1, 2]$, $y \in [-1, 2]$, $z \in [-2, -1]$.

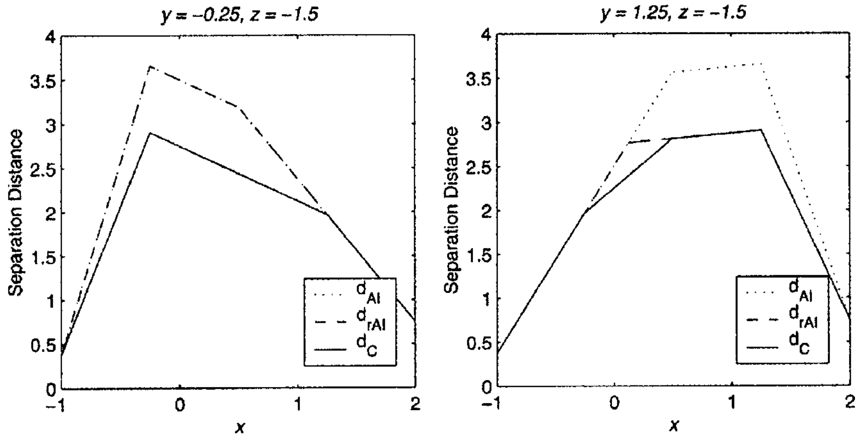

Figure 7. Separation distances for the convex envelope, arithmetic interval and recursive arithmetic interval underestimators $x \in [-1, 2]$, $y \in [-1, 2]$, $z \in [-2, -1]$.

6.2. ARITHMETIC INTERVALS

The arithmetic interval (AI) scheme is based on the multiplication of variable bounding inequalities followed by the linearization of these constraints through variable substitutions (Hamed. 1991). In the case of a trilinear monomial the AI scheme is based the on the inequalities,

$$(x - \underline{x})(y - \underline{y})(z - \underline{z}) \geq 0,$$
$$(\overline{x} - x)(\overline{y} - y)(z - \underline{z}) \geq 0,$$
$$(\overline{x} - x)(y - \underline{y})(\overline{z} - z) \geq 0,$$
$$(x - \underline{x})(\overline{y} - y)(\overline{z} - z) \geq 0.$$

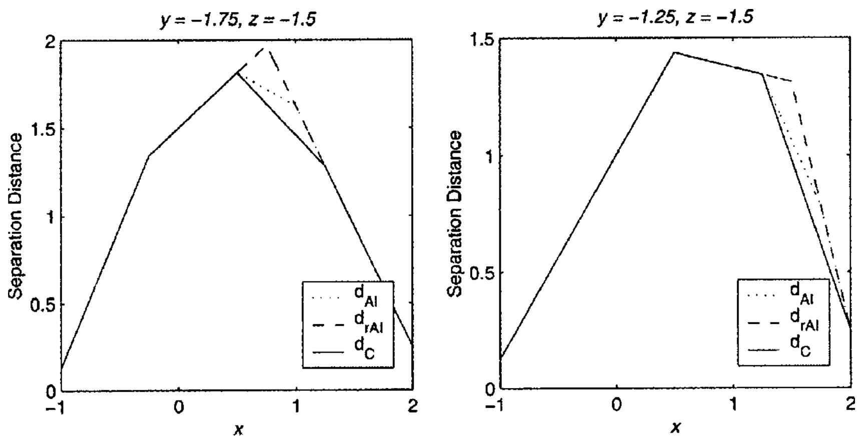

Figure 8. Separation distances for the convex envelope, arithmetic interval and recursive arithmetic interval underestimators $x \in [-1, 2]$, $y \in [-2, -1]$, $z \in [-2, -1]$.

Table II. Lower bounding function values for $\mathbf{x} \in [1,2] \times [-1,2] \times [-2,1]$

(x, y, z)	xyz	f_{AI}	f_{rAI}	$\underline{f}$
(1.000,−0.25,−0.5)	0.1250	−1.000	−1.00	−1.000
(1.125,−0.25,−0.5)	0.1406	−1.875	−1.50	−1.500
(1.250,−0.25,−0.5)	0.1563	−2.750	−2.00	−2.000
(1.375,−0.25,−0.5)	0.1719	−2.875	−2.50	−2.125
(1.500,−0.25,−0.5)	0.1875	−3.000	−3.00	−2.250
(1.625,−0.25,−0.5)	0.2032	−2.750	−2.75	−2.375
(1.750,−0.25,−0.5)	0.2188	−2.500	−2.50	−2.500

In the general case four variables are introduced, w_{xyz}, w_{xy}, w_{xz}, and w_{yz}, as substitutes for xyz, xy, xz, and yz. w_{xyz} is bounded from below by the system of inequalities,

$$
\begin{aligned}
w_{xyz} &\geq \underline{z}w_{xy} + \underline{y}w_{xz} + \underline{x}w_{yz} - \underline{y}\underline{z}x - \underline{x}\underline{z}y - \underline{x}\underline{y}z + \underline{x}\underline{y}\underline{z}, \\
w_{xyz} &\geq \underline{z}w_{xy} + \overline{y}w_{xz} + \overline{x}w_{yz} - \overline{y}\underline{z}x - \overline{x}\underline{z}y - \overline{x}\overline{y}z + \overline{x}\overline{y}\underline{z}, \\
w_{xyz} &\geq \overline{z}w_{xy} + \underline{y}w_{xz} + \overline{x}w_{yz} - \underline{y}\overline{z}x - \overline{x}\overline{z}y - \overline{x}\underline{y}z + \overline{x}\underline{y}\overline{z}, \\
w_{xyz} &\geq \overline{z}w_{xy} + \overline{y}w_{xz} + \underline{x}w_{yz} - \overline{y}\overline{z}x - \underline{x}\overline{z}y - \underline{x}\overline{y}z + \underline{x}\overline{y}\overline{z},
\end{aligned}
\tag{6.29}
$$

and the inequalities (6.28) constraining the bilinear variables, w_{xy}, w_{xz}, and w_{yz}.

A lower bounding function $f_{\mathrm{AI}}(x, y, z)$ is defined as follows:

$$
f_{\mathrm{AI}}(x, y, z) = \min_{w_{xyz}, w_{xy}, w_{xz}, w_{yz}} w_{xyz}
$$

$$
\text{subject to (6.29) and (6.28)}
$$

Note that the reformulation-linearization technique of Sherali and Tuncbilek (1992) is a generalization of this convexification strategy.

6.3. SEPARATION DISTANCE COMPARISONS

In this section the separation distances between xyz and the rAI and AI underestimators are compared with the separation distance between xyz and the convex envelope. The separation distances between the function xyz and the lower bounding functions $f_{\mathrm{AI}}(x, y, z)$ and $f_{\mathrm{rAI}}(x, y, z)$ are defined as

$$
d_{\mathrm{AI}}(x, y, z) := xyz - f_{\mathrm{AI}}(x, y, z),
$$

and

$$
d_{\mathrm{rAI}}(x, y, z) := xyz - f_{\mathrm{rAI}}(x, y, z).
$$

In Figures 3 to 8 two graphs are presented for each sign combination. In each graph y and z are constant, while the separation distances are plotted as a function of x.

In Figure 5, the AI and rAI systems are shown to generate poor bounds relative to the convex envelope. As the form of the AI and rAI constraints do not match the form of the constraints that result from the application of Lemmas 2.1 and 2.2 the AI and rAI bounding schemes cannot, in general, represent these types of facet.

Fixing the signs on the lower and upper bounds, $\underline{x}, \underline{y}, \underline{z}, \overline{x}, \overline{y}$ and $\overline{z}$, it is possible to eliminate the variables w_{xy}, w_{xz} and w_{yz} from the systems of constraints (6.29) and (6.27). For example, when $\underline{x} \geq 0, \underline{y} \geq 0$, $\underline{z} < 0$ and $\overline{z} \geq 0$ the rAI scheme can be represented by the following system of constraints:

$$
\begin{aligned}
w_{xyz} &\geq \overline{y}\overline{z}x + \overline{x}\overline{z}y + \overline{x}\overline{y}z - 2\overline{x}\overline{y}\overline{z},\\
w_{xyz} &\geq \overline{y}\underline{z}x + \underline{x}\overline{z}y + \underline{x}\overline{y}z - \underline{x}\overline{y}\overline{z} - \underline{x}\overline{y}\underline{z},\\
w_{xyz} &\geq \overline{y}\underline{z}x + \underline{x}\underline{z}y + \underline{x}\underline{y}z - \underline{x}\underline{y}\underline{z} - \underline{x}\overline{y}\underline{z},\\
w_{xyz} &\geq \underline{y}\overline{z}x + \overline{x}\underline{z}y + \overline{x}\underline{y}z - \overline{x}\underline{y}\overline{z} - \overline{x}\underline{y}\underline{z},\\
w_{xyz} &\geq \underline{y}\underline{z}x + \overline{x}\underline{z}y + \underline{x}\underline{y}z - \underline{x}\underline{y}\underline{z} - \overline{x}\underline{y}\underline{z},\\
w_{xyz} &\geq \overline{y}\underline{z}x + \overline{x}\overline{z}y + \underline{x}\overline{y}z - \underline{x}\overline{y}\underline{z} - \overline{x}\overline{y}\overline{z},\\
w_{xyz} &\geq \overline{y}\underline{z}x + \underline{x}\underline{z}y + \underline{x}\underline{y}z - \underline{x}\overline{y}\underline{z} - \underline{x}\underline{y}\underline{z},\\
w_{xyz} &\geq \underline{y}\underline{z}x + \overline{x}\underline{z}y + \underline{x}\underline{y}z - \overline{x}\overline{y}\underline{z} - \underline{x}\underline{y}\underline{z},\\
w_{xyz} &\geq \overline{y}\overline{z}x + \overline{x}\underline{z}y + \overline{x}\underline{y}z - \overline{x}\underline{y}\underline{z} - \overline{x}\overline{y}\overline{z},\\
w_{xyz} &\geq \underline{y}\overline{z}x + \underline{x}\overline{z}y + \overline{x}\overline{y}z - \underline{x}\underline{y}\overline{z} - \overline{x}\overline{y}\overline{z}.
\end{aligned}
$$

Note that each coefficient is a product of two bounds, the same structure as that seen in the constraints generated by Lemmas 2.3 and 2.4. The first five inequalities define halfspaces supported by facets from Equations (3.19), (3.20), (3.21), (3.22), and (3.23) respectively. Although their structure is similar, the rAI and AI schemes, do not always generate constraints that match the constraints from Lemmas 2.3 and 2.4.

Lower bounding function values in $\mathbf{x} = [1, 2] \times [-1, 2] \times [-2, 1]$ are listed in Table II . In this table the value on the convex envelope is denoted $\underline{f}$.

6.4. RIKUN'S FACETS

Rikun (1997) proposed a formula that may define some elements of the convex envelope of a multilinear function. For the trilinear monomial this formula is as follows:

$$w = \xi_2\xi_3x_1 + \xi_1\xi_3x_2 + \xi_1\xi_2x_3 - 2\xi_1\xi_2\xi_3 \tag{6.30}$$

where $\xi = (\xi_1, \xi_2, \xi_3)$ is any point in $\Re^3$. A facet of $\underline{\mathcal{C}}_3(\mathbf{x})$ may be obtained from Equation (6.30) if for a given ξ, w underestimates $x_1x_2x_3$ at all vertices of the hyper-rectangle $\mathbf{x}$. For example, let $\mathbf{x} = [-1, 2] \times [-5, -2] \times [-3, 5]$ as in Illustration 1. We use the vertices of $\mathbf{x}$ as values of ξ. From $\xi = (-1, -5, 5)$ we get the equation:

$$w = -25x_1 - 5x_2 + 5x_3 - 50.$$

This is valid at all vertices of $\mathbf{x}$, and therefore defines a facet of $\underline{\mathcal{C}}_3(\mathbf{x})$. Similarly from $\xi = (2, -5, -3)$ we obtain the facet defining equality:

$$w = 15x_1 - 6x_2 - 10x_3 - 60.$$

Any of the other vertices used for ξ yield non-underestimators. In Illustration 1 we saw that there are six facets of $\underline{\mathcal{C}}_3(\mathbf{x})$, the above two and four others which cannot be obtained from Equation (6.30).

7. Conclusions

In the paper we have presented a complete and explicit description of the facets of the convex and concave envelopes for trilinear functions for the cases where the signs on the variable bounds are different for at least one variable. The result is useful from a computational point of view as a convex linear lower bounding or upper bounding relaxation of a trilinear term can be established using only five or six linear inequalities, and introducing only a single new variable. The separation distance between the trilinear function and the convex envelope was compared with that of approximations of the convex envelope, namely the arithmetic interval and recursive arithmetic interval approximations. In most instances the approximations were found to be weaker than the convex envelope itself.

References

1. Adjiman, C.S., Androulakis, I.P. and Floudas, C.A. (1998b), A global optimization method, α BB, for general twice differentiable NLPs. I. Implementation and computional results, *Computers and Chemical Engineering* 22, 1159–1179.
2. Adjiman, C.S., S. Dallwig, C.A. Floudas, and A. Neumaier (1998a), A global optimization method, αBB, for general twice differentiable NLPs. I. Theoretical advances, *Computers and Chemical Engineering* 22, 1137–1158.
3. Al-Khayyal, F.A. and J.E. Falk (1983), Jointly constrained biconvex programming. *Mathematics Operation Research* 8, 273.
4. Androulakis, I.P., C.D. Maranas, and C.A. Floudas (1995), αBB: a global optimization method for general constrained nonconvex problems. *Journal. of Global Optimization* 7, 337–363.
5. Balas, E. and J. Mazzola (1984), Nonlinear 0–1 Programming: I Linearization techniques. *Mathematical Programming* 30, 1–21.
6. Crama, Y. (1989), Recognition problems for special classes of polynomials in 0-1 variables. *Mathematical Programming* 44, 139–155.
7. Crama, Y. (1993), Concave extensions for nonlinear 0–1 maximization problems. *Mathematical Programming* 61, 53–60.
8. Floudas, C.A. (2000), *Deterministic Global Optimization: Theory, Algorithms and Applications*. Kluwer Academic Publishers.

9. Floudas, C.A., A. Aggarwal and A.R. Ciric 1989, Global Optimum Search for Nonconvex NLP and MINLP Problems. *Computers and Chemical Engineering* 13, 1117–1132.
10. Glover, F. and E. Woolsey (1974), Converting the 0-1 polynomial programming problem to a linear 0-1 program. *Operations Research* 22, 180–182.
11. Hamed, A. S.E. (1991), Calculation of Bounds on Variables and Underestimating Convex Functions for Nonconvex Functions, Ph.D. thesis, George Washington University.
12. Hammer, P., P. Hansen and B. Simeone (1984), Roof duality, complementation and persistency in quadratic 0-1 optimization. *Mathematical Programming* 28, 121–155.
13. Hammer, P. and S. Rudeanu (1968), *Boolean Methods in Operations Research and Related Areas*. Springer.
14. Hansen, P. and B. Simeone (1990), On the equivalence of paved duality and standard linearization in nonlinear 0-1 optimization. *Discrete Applied Mathematics* 29, 187–193.
15. Maranas, C.D. and C.A. Floudas (1995), Finding all Solutions of Nonlinearly Constrained Systems of Equations, *Journal. of Global Optimization*. 7, 143–182.
16. McCormick, G.P. (1976), Computation of Global Solutions to Factorable Nonconvex Programs: Part I-Convex Underestimation Problems. *Mathematical Programming* 10, 147.
17. Meyer, C.A. and C.A. Floudas (2003), Trilinear Monomials with Positive or Negative Domains: Facets of the Convex and Concave Envelopes, *to appear in Frontiers in Global Optimization*, C.A. Floudas and P.M. Pardolos (eds.), Kluwer Academic Publishers.
18. Rikun, A.D. (1997), A Convex Envelope Formula for Multilinear Functions. *Journal. of Global Optimization*. 10, 425–437.
19. Rockafellar, R.T. (1970), *Convex Analysis*. Princeton University Press.
20. Ryoo, H.S. and N.V. Sahinidis (2001), Analysis of Bounds for Multilinear Functions. *Journal of Global Optimization* 19, 403–424.
21. Sherali, H.D. 1997, Convex Envelopes of Multilinear Functions over a Unit Hypercube and over Special Discrete Sets. *Acta Mathematica Vietnamica* 22, 245–270.
22. Sherali, H.D. and C.H. Tuncbilek (1992), A Global Optimization Algorithm for Polynomial Programming Problems Using a Reformulation-Linearization Technique. *Jornal. of Global Optimization*. 2, 101–112.
23. Tawarmalani, M. and N.V. Sahinidis (2002), Convex extensions and envelopes of lower semi-continuous functions. *Mathematical. Programming Ser. A* 1.